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TEMPORAL DERIVATIVES IN THE FINITE-ELEMENT METHOD ON CONTINUOUSLY DEFORMING GRIDS*

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Abstract. When solving certain time-dependent partial differential equations using a finite-element technique on a deforming grid, it is shown that there is a need to differentiate the trial solution with respect to the position of each of the moveable node points. A result is presented which enables these nonstandard derivatives to be expressed in terms of the standard spatial derivatives of the trial functions provided that certain conditions are satisfied. These conditions are investigated and shown to be satisfied for a large class of trial spaces. Even when the necessary conditions are not satisfied by all of the trial functions being used, the result is shown to be still of great use in evaluating these awkward nodal derivatives.

Key words. time-dependent partial differential equations, finite-element method, moving grids

AMS(MOS) subject classification. 65M60

1. Introduction. In this paper we consider the use of the finite-element method to solve a general time-dependent partial differential equation of the form

$$\frac{\partial u}{\partial t}(\mathbf{x}, t) = \mathcal{L}(u(\mathbf{x}, t)), \tag{1.1}$$

where $\mathcal{L}$ is an operator depending on $\mathbf{x}$, $u(\mathbf{x}, t)$ and its spatial derivatives, and there are appropriate initial and boundary conditions. The solution u may be scalar or vector-valued. Many problems of this type have solutions which can most efficiently be represented on a finite-element mesh which deforms continuously in time. In such cases it is often appropriate to employ a solution procedure which uses a moving grid. For examples of this see Bonnerot and Jamet [1], [2], Lynch et al. [5]-[8], [14], Miller et al. [4], [9], [10], or Mosher [12].

When a continuously deforming grid is used, the finite-element trial solution is of the form

$$v(\mathbf{x}, t) = \sum_{i=1}^{N} a_i(t)\alpha_i(\mathbf{x}, \mathbf{s}(t)), \tag{1.2}$$

where Span $\{\alpha_i(\mathbf{x}, \mathbf{s}(t)): i = 1, \cdots, N\}$ is the trial space and $\mathbf{s}(t)$ is an ordered set of the position vectors of the node points of the grid which we will assume to be made up of simplexes. Each trial function α_i is time-dependent due to the motion of the grid.

At some stage in the solution procedure, it will be necessary to differentiate v with respect to time, giving an expression of the form

$$\frac{\partial v}{\partial t} = \sum_{i=1}^{N} \left\{ \dot{a}_i(t)\alpha_i(\mathbf{x}, \mathbf{s}(t)) + a_i(t)\frac{\partial \alpha_i}{\partial \mathbf{s}}(\mathbf{x}, \mathbf{s}(t)) \cdot \dot{\mathbf{s}}(t) \right\}, \tag{1.3}$$

where the dot above a letter represents its temporal derivative. So we see that for each node, $j = 1, \cdots, M$, say with position $\mathbf{s}_j$, it will be necessary to evaluate the terms

$$\frac{\partial \alpha_i}{\partial \mathbf{s}_j}(\mathbf{x}, \mathbf{s}(t)) = \left(\frac{\partial \alpha_i}{\partial s_{j,1}}, \cdots, \frac{\partial \alpha_i}{\partial s_{j,n}}\right)^T(\mathbf{x}, \mathbf{s}(t)), \tag{1.4}$$

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where, in the above notation, $\mathbf{x} \in \mathbb{R}^n$,

$$s_{j,d} = \mathbf{s}_j \cdot \hat{\mathbf{e}}_d, \qquad d = 1, \cdots, n, \tag{1.5}$$

and $\hat{\mathbf{e}}_d$ is the unit vector in the dth coordinate direction.

In fact, (1.3) could be written as

$$\frac{\partial v}{\partial t} = \sum_{i=1}^{N} \{\dot{a}_i(t)\alpha_i(\mathbf{x}, \mathbf{s}(t))\} + \frac{\partial v}{\partial \mathbf{s}}(\mathbf{x}, \mathbf{s}(t)) \cdot \dot{\mathbf{s}}(t), \tag{1.6}$$

which means that we are ultimately interested in the derivatives of v with respect to each $\mathbf{s}_j$. It is the evaluation of these derivatives $\partial v/\partial \mathbf{s}_j$ (and so $\partial \alpha_i/\partial \mathbf{s}_j$) with which we will be concerned in this paper.

The principal result that we shall rigorously establish is a simple formula for $\partial \alpha_i/\partial \mathbf{s}_j$. This formula is stated in [5] and [6]. We prove that if, in addition to some technical hypotheses, on each simplex, we can write α_i strictly in terms of the piecewise linear Lagrange basis functions which are nonzero on that simplex, then $\partial \alpha_i/\partial \mathbf{s}_j$ can be written simply in terms of the spatial derivatives of α_i. For any Lagrange finite-element basis this condition is satisfied. We further show that the formula is of use for Hermite elements, and is easily amended for basis functions not satisfying the condition above.

2. The main result. In this section we will prove a theorem which is based on an idea of Lynch and Gray [6] and Lynch [5]. The result will be more rigorously presented here than in their original papers, and so its limitations and difficulties will become more apparent. Before deriving this result it will be useful to make the following definition.

DEFINITION. For a given problem, a grid of simplexes $\mathbf{s}(t)$ will be said to be allowable if it has the same connectivity as the initial grid $\mathbf{s}(0)$, and can be obtained from it by a continuous deformation in which each element of the grid always has strictly positive measure.

Next we prove three lemmas which will be helpful in both the statement and the proof of the main theorem.

Lemma 2.1 gives a relation between $\mathbf{x}$, a position anywhere within the grid at time t, and $\boldsymbol{\xi}$, a position within some fixed reference grid. We choose as this reference grid the mesh at time $t = 0$. This way it is guaranteed that the present grid and the reference grid will always be topologically equivalent. It is then shown that the given relation forms a one-to-one correspondence between the two domains, hence it implicitly defines a function $\boldsymbol{\xi}$ of the variables $\mathbf{x}$ and $\mathbf{s}$.

LEMMA 2.1. *On an allowable grid of simplexes in $\mathbb{R}^n$, with N vertices, $\mathbf{s}_1, \cdots, \mathbf{s}_N$ say, the expression*

$$\mathbf{x} = \sum_{k=1}^{N} \mathbf{s}_k(t)\psi_k(\boldsymbol{\xi}, \mathbf{s}(0)) \tag{2.1}$$

implicitly defines a function $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x}, \mathbf{s}(t))$, where ψ_k is the linear hat function taking value 1 at $\mathbf{s}_k(t)$.

Proof. Choose an arbitrary element of the grid at time t, $\eta(t)$, say, with vertices at $\mathbf{s}_{e(0)}(t), \cdots, \mathbf{s}_{e(n)}(t)$, say. Then, on this element, (2.1) becomes

$$\mathbf{x} = \sum_{k=0}^{n} \mathbf{s}_{e(k)}(t)\psi_{e(k)}(\boldsymbol{\xi}, \mathbf{s}(0)). \tag{2.2}$$

Now, given any $\mathbf{x} \in \eta(t)$, there exists a unique

$$\mathbf{a} \in \left\{ \mathbb{R}^{n+1} : \left(\sum_{k=0}^{n} a_k \right) = 1 \right\} \tag{2.3}$$

such that

$$\mathbf{x} = \sum_{k=0}^{n} a_k \mathbf{s}_{e(k)}(t). \tag{2.4}$$

The a_k's are the barycentric coordinates of the point $\mathbf{x}$ with respect to the $\mathbf{s}_{e(k)}(t)$'s. From (2.2) we see that

$$\psi_{e(k)}(\boldsymbol{\xi}, \mathbf{s}(0)) = a_k, \qquad k = 0, \cdots, n. \tag{2.5}$$

However, it is a simple property of the linear hat functions that

$$\boldsymbol{\xi} = \sum_{k=0}^{n} \mathbf{s}_{e(k)}(0) \psi_{e(k)}(\boldsymbol{\xi}, \mathbf{s}(0)), \tag{2.6}$$

and so (2.5) means that we have

$$\boldsymbol{\xi} = \sum_{k=0}^{n} a_k \mathbf{s}_{e(k)}(0), \tag{2.7}$$

which is a uniquely defined point in $\eta(0)$, depending only on $\mathbf{x}$. Thus, (2.1) defines a (one-to-one) mapping from $\eta(t)$ to $\eta(0)$. This is true for each simplex element of the grid $\mathbf{s}(t)$, and so the result follows. □

Lemma 2.2 expands a little more on the above result, giving an explicit form of the inverse to equation (2.1).

LEMMA 2.2. *On any allowable grid of simplexes in* $\mathbb{R}^n$, *with* $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x}, \mathbf{s}(t))$ *defined by* (2.1),

$$\boldsymbol{\xi} = \sum_{k=1}^{N} \mathbf{s}_k(0) \psi_k(\mathbf{x}, \mathbf{s}(t)). \tag{2.8}$$

Proof. Again, choose an arbitrary element, $\eta(t)$ say, of the grid, with vertices at $\mathbf{s}_{e(0)}(t), \cdots, \mathbf{s}_{e(n)}(t)$, say. Then on this element (2.1) becomes

$$\mathbf{x} = \sum_{k=0}^{n} \mathbf{s}_{e(k)}(t) \psi_{e(k)}(\boldsymbol{\xi}, \mathbf{s}(0)). \tag{2.9}$$

Also, as for (2.6), we can say

$$\boldsymbol{\xi} = \sum_{k=0}^{n} \mathbf{s}_{e(k)}(0) \psi_{e(k)}(\boldsymbol{\xi}, \mathbf{s}(0)) \tag{2.10}$$

and

$$\mathbf{x} = \sum_{k=0}^{n} \mathbf{s}_{e(k)}(t) \psi_{e(k)}(\mathbf{x}, \mathbf{s}(t)). \tag{2.11}$$

Now, since

$$\sum_{k=0}^{n} \psi_{e(k)}(\boldsymbol{\xi}, \mathbf{s}(0)) \equiv \sum_{k=0}^{n} \psi_{e(k)}(\mathbf{x}, \mathbf{s}(t)) \equiv 1, \tag{2.12}$$

the uniqueness of the barycentric coordinates of $\mathbf{x}$ with respect to the $\mathbf{s}_{e(k)}(t)$'s, along with equations (2.9) and (2.11), imply that

$$\psi_{e(k)}(\boldsymbol{\xi}, \mathbf{s}(0)) = \psi_{e(k)}(\mathbf{x}, \mathbf{s}(t)), \qquad k = 0, \cdots, n. \tag{2.13}$$

Now, (2.10) and (2.13) give the required result on $\eta(t)$. But this is an arbitrary element, and so the result must also hold on all other such elements, thus giving equation (2.8). □

The next result contains most of the work required for the proof of Theorem 2.4 which follows it. The lemma provides a relationship between the nonstandard derivative $\partial\psi_i/\partial\mathbf{s}_j$ and the conventional spatial derivative $\partial\psi_i/\partial\mathbf{x}$, where ψ_i is the usual linear hat function which takes value 1 at $\mathbf{s}_i$.

LEMMA 2.3. *On any allowable grid of simplexes in* $\mathbb{R}^n$,

$$\frac{\partial\psi_i}{\partial\mathbf{s}_j}(\mathbf{x},\mathbf{s}(t)) = -\psi_j(\mathbf{x},\mathbf{s}(t))\frac{\partial\psi_i}{\partial\mathbf{x}}(\mathbf{x},\mathbf{s}(t)), \tag{2.14}$$

where ψ_k *is the linear hat function taking value* 1 *at* $\mathbf{s}_k(t)$.

Proof. Once more, choose an arbitrary element $\eta(t)$ with vertices at $\mathbf{s}_{e(0)}(t),\cdots,\mathbf{s}_{e(n)}(t)$. From (2.11) we have

$$\mathbf{x} = \sum_{k=0}^{n}\mathbf{s}_{e(k)}(t)\psi_{e(k)}(\mathbf{x},\mathbf{s}(t)). \tag{2.15}$$

Differentiating this with respect to x_d yields

$$\hat{\mathbf{e}}_d = \sum_{k=0}^{n}\mathbf{s}_{e(k)}(t)\frac{\partial\psi_{e(k)}}{\partial x_d}(\mathbf{x},\mathbf{s}(t)), \tag{2.16}$$

and differentiating (2.15) with respect to $s_{e(0),d}$ $(=\mathbf{s}_{e(0)}\cdot\hat{\mathbf{e}}_d)$ yields

$$\mathbf{0} = \sum_{k=0}^{n}\left\{\mathbf{s}_{e(k)}(t)\frac{\partial\psi_{e(k)}}{\partial s_{e(0),d}}(\mathbf{x},\mathbf{s}(t))\right\} + \hat{\mathbf{e}}_d\psi_{e(0)}(x,s(t)). \tag{2.17}$$

Substituting (2.16) into (2.17) gives

$$\mathbf{0} = \sum_{k=0}^{n}\mathbf{s}_{e(k)}(t)\left\{\frac{\partial\psi_{e(k)}}{\partial s_{e(0),d}}(\mathbf{x},\mathbf{s}(t)) + \psi_{e(0)}(\mathbf{x},\mathbf{s}(t))\frac{\partial\psi_{e(k)}}{\partial x_d}(\mathbf{x},\mathbf{s}(t))\right\}, \tag{2.18}$$

which can be written as

$$\begin{aligned}\mathbf{0} = \mathbf{s}_{e(0)}&\sum_{k=0}^{n}\left\{\frac{\partial\psi_{e(k)}}{\partial s_{e(0),d}}(\mathbf{x},\mathbf{s}(t)) + \psi_{e(0)}(\mathbf{x},\mathbf{s}(t))\frac{\partial\psi_{e(k)}}{\partial x_d}(\mathbf{x},\mathbf{s}(t))\right\}\\ &+\sum_{k=1}^{n}(\mathbf{s}_{e(k)}(t)-\mathbf{s}_{e(0)}(t))\\ &\cdot\left\{\frac{\partial\psi_{e(k)}}{\partial s_{e(0),d}}(\mathbf{x},\mathbf{s}(t)) + \psi_{e(0)}(\mathbf{x},\mathbf{s}(t))\frac{\partial\psi_{e(k)}}{\partial x_d}(\mathbf{x},\mathbf{s}(t))\right\}.\end{aligned} \tag{2.19}$$

But,

$$\sum_{k=0}^{n}\psi_{e(k)}(\mathbf{x},\mathbf{s}(t)) \equiv 1, \tag{2.20}$$

and so

$$\sum_{k=0}^{n}\left\{\frac{\partial\psi_{e(k)}}{\partial s_{e(0),d}}(\mathbf{x},\mathbf{s}(t)) + \psi_{e(0)}(\mathbf{x},\mathbf{s}(t))\frac{\partial\psi_{e(k)}}{\partial x_d}(\mathbf{x},\mathbf{s}(t))\right\} \equiv 0. \tag{2.21}$$

Also, $\{(\mathbf{s}_{e(k)}(t)-\mathbf{s}_{e(0)}(t)): k=1,\cdots,n\}$ is linearly independent, so (2.19) tells us that

$$\frac{\partial\psi_{e(k)}}{\partial s_{e(0),d}}(\mathbf{x},\mathbf{s}(t)) = -\psi_{e(0)}(\mathbf{x},\mathbf{s}(t))\frac{\partial\psi_{e(k)}}{\partial x_d}(\mathbf{x},\mathbf{s}(t)), \qquad k=1,\cdots,n, \tag{2.22}$$

and from (2.20) and (2.22) we can also deduce that

$$\frac{\partial \psi_{e(0)}}{\partial s_{e(0),d}}(\mathbf{x}, \mathbf{s}(t)) = -\psi_{e(0)}(\mathbf{x}, \mathbf{s}(t)) \frac{\partial \psi_{e(0)}}{\partial x_d}(\mathbf{x}, \mathbf{s}(t)). \tag{2.23}$$

This is all true for any local numbering of the element; therefore (2.22) and (2.23) can be generalized to

$$\frac{\partial \psi_{e(k)}}{\partial s_{e(j),d}}(\mathbf{x}, \mathbf{s}(t)) = -\psi_{e(j)}(\mathbf{x}, \mathbf{s}(t)) \frac{\partial \psi_{e(k)}}{\partial x_d}(\mathbf{x}, \mathbf{s}(t)), \qquad k = 0, \cdots, n, \quad j = 0, \cdots, n. \tag{2.24}$$

By noting that both sides of (2.24) are zero on $\eta(t)$ when $\mathbf{s}_j \notin \{\mathbf{s}_{e(k)}: k = 0, \cdots, n\}$, we can write this as

$$\frac{\partial \psi_{e(k)}}{\partial s_{j,d}}(\mathbf{x}, \mathbf{s}(t)) = -\psi_j(\mathbf{x}, \mathbf{s}(t)) \frac{\partial \psi_{e(k)}}{\partial x_d}(\mathbf{x}, \mathbf{s}(t)), \qquad k = 0, \cdots, n, \quad j = 1, \cdots, N. \tag{2.25}$$

Finally, this result holds on each element of the grid and so, in global notation, it becomes

$$\frac{\partial \psi_i}{\partial \mathbf{s}_j}(\mathbf{x}, \mathbf{s}(t)) = -\psi_j(\mathbf{x}, \mathbf{s}(t)) \frac{\partial \psi_i}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{s}(t)), \qquad i, j = 1, \cdots, N, \tag{2.26}$$

and the lemma is proved. □

We are now in a position to prove the main result of this paper. It is an extension of Lemma 2.3 to a wider class of basis functions—α_i, say. Such basis functions must have the property that there exists a corresponding function—$\bar{\alpha}_i$, say—such that for every $\mathbf{x}$ in the domain at time t,

$$\alpha_i(\mathbf{x}, \mathbf{s}(t)) = \bar{\alpha}_i(\boldsymbol{\xi}(\mathbf{x}, \mathbf{s}(t))),$$

where $\boldsymbol{\xi}$ is given by (2.1). Consequently, α_i must be such that for all $\mathbf{x}_1$ within the grid at time t_1, with corresponding $\mathbf{x}_2$ at time t_2 (i.e., $\boldsymbol{\xi}(\mathbf{x}_1, \mathbf{s}(t_1)) = \boldsymbol{\xi}(\mathbf{x}_2, \mathbf{s}(t_2))$), we have

$$\alpha_i(\mathbf{x}_1, \mathbf{s}(t_1)) = \alpha_i(\mathbf{x}_2, \mathbf{s}(t_2)).$$

In words, this says that as the mesh and the basis functions deform with time, each point within a simplex element moves such that α_i evaluated at the current position of the point remains the same.

THEOREM 2.4. *Suppose* $\alpha_i(\mathbf{x}, \mathbf{s}(t))$ *is any finite-element basis function defined on an allowable grid of simplexes in* $\mathbb{R}^n$. *Again define* $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x}, \mathbf{s}(t))$ *by*

$$\mathbf{x} = \sum_{k=1}^{N} \mathbf{s}_k(t) \psi_k(\boldsymbol{\xi}, \mathbf{s}(0)). \tag{2.27}$$

Then, if there exists an $\bar{\alpha}_i: \mathbb{R}^n \to \mathbb{R}$ *such that*

$$\alpha_i(\mathbf{x}, \mathbf{s}(t)) \equiv \bar{\alpha}_i(\boldsymbol{\xi}(\mathbf{x}, \mathbf{s}(t))), \tag{2.28}$$

it follows that

$$\frac{\partial \alpha_i}{\partial \mathbf{s}_j}(\mathbf{x}, \mathbf{s}(t)) = -\psi_j(\mathbf{x}, \mathbf{s}(t)) \frac{\partial \alpha_i}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{s}(t)). \tag{2.29}$$

Proof. From condition (2.28) we can write

$$\frac{\partial \alpha_i}{\partial s_{j,d}}(\mathbf{x}, \mathbf{s}(t)) = \frac{d\bar{\alpha}_i}{d\boldsymbol{\xi}}(\boldsymbol{\xi}) \cdot \frac{\partial \boldsymbol{\xi}}{\partial s_{j,d}}(\mathbf{x}, \mathbf{s}(t)), \qquad d = 1, \cdots, n, \tag{2.30}$$

and

$$\frac{\partial \alpha_i}{\partial x_d}(\mathbf{x}, \mathbf{s}(t)) = \frac{d\bar{\alpha}_i}{d\boldsymbol{\xi}}(\boldsymbol{\xi}) \cdot \frac{\partial \boldsymbol{\xi}}{\partial x_d}(\mathbf{x}, \mathbf{s}(t)), \qquad d = 1, \cdots, n. \tag{2.31}$$

Using Lemma 2.2, these become

$$\frac{\partial \alpha_i}{\partial s_{j,d}}(\mathbf{x}, \mathbf{s}(t)) = \frac{d\bar{\alpha}_i}{d\boldsymbol{\xi}}(\boldsymbol{\xi}) \cdot \left[\sum_{k=1}^{N} \mathbf{s}_k(0) \frac{\partial \psi_k}{\partial s_{j,d}}(\mathbf{x}, \mathbf{s}(t)) \right], \qquad d = 1, \cdots, n, \tag{2.32}$$

and

$$\frac{\partial \alpha_i}{\partial x_d}(\mathbf{x}, \mathbf{s}(t)) = \frac{d\bar{\alpha}_i}{d\boldsymbol{\xi}}(\boldsymbol{\xi}) \cdot \left[\sum_{k=1}^{N} \mathbf{s}_k(0) \frac{\partial \psi_k}{\partial x_d}(\mathbf{x}, \mathbf{s}(t)) \right], \qquad d = 1, \cdots, n. \tag{2.33}$$

Lemma 2.3 applied to (2.32) gives

$$\frac{\partial \alpha_i}{\partial s_{j,d}}(\mathbf{x}, \mathbf{s}(t)) = \frac{d\bar{\alpha}_i}{d\boldsymbol{\xi}}(\boldsymbol{\xi}) \cdot \left[-\psi_j(\mathbf{x}, \mathbf{s}(t)) \sum_{k=1}^{N} \mathbf{s}_k(0) \frac{\partial \psi_k}{\partial x_d}(\mathbf{x}, \mathbf{s}(t)) \right], \qquad d = 1, \cdots, n; \tag{2.34}$$

thus, (2.33) and (2.34) give

$$\frac{\partial \alpha_i}{\partial s_{j,d}}(\mathbf{x}, \mathbf{s}(t)) = -\psi_j(\mathbf{x}, \mathbf{s}(t)) \frac{\partial \alpha_i}{\partial x_d}(\mathbf{x}, \mathbf{s}(t)), \qquad d = 1, \cdots, n,$$

as required. □

The theorem above therefore allows us to write the nonstandard derivatives with respect to nodal movement in terms of the usual spatial derivatives of the basis functions, provided that they satisfy condition (2.28). This is a particularly useful relation since the latter derivatives will need to be calculated anyway in order to evaluate an approximation to $\mathscr{L}(v)$. Hence, no extra work is introduced into the finite-element procedure by the appearance of these derivatives with respect to nodal position in the expression for $\partial v/\partial t$ (1.3).

It is clear that not all finite-element basis functions actually do satisfy condition (2.28). A simple example of one that does not comes from the usual basis for Hermite cubic functions in one dimension. This basis will be considered in some detail in § 4.

3. Lagrange finite elements. We now need to try and establish exactly which finite-element basis functions do and do not satisfy the constraint (2.28). In order to do this we will attempt to gain a better understanding of what this particular condition really means.

LEMMA 3.1. *Given a finite-element basis function* $\alpha_j(\mathbf{x}, \mathbf{s}(t))$ *on some allowable grid of simplexes in* $\mathbb{R}^n$, *with* $\boldsymbol{\xi}$ *defined by* (2.1), *the following three statements are equivalent*:

(a) *There exists* $\bar{\alpha}_j : \mathbb{R}^n \to \mathbb{R}$ *such that* $\alpha_j(\mathbf{x}, \mathbf{s}(t)) \equiv \bar{\alpha}_j(\boldsymbol{\xi}(\mathbf{x}, \mathbf{s}(t)))$.

(b) *There exists* $f_j : \mathbb{R}^N \to \mathbb{R}$ *such that*

$$\alpha_j(\mathbf{x}, \mathbf{s}(t)) \equiv f_j(\psi_1(\mathbf{x}, \mathbf{s}(t)), \cdots, \psi_N(\mathbf{x}, \mathbf{s}(t))).$$

(c) $\alpha_j(\mathbf{x}, \mathbf{s}(t)) \equiv \alpha_j(\boldsymbol{\xi}(\mathbf{x}, \mathbf{s}(t)), \mathbf{s}(0))$.

Moreover, they all imply that

(d) *The range of values taken by* $\alpha_j(\mathbf{x}, \mathbf{s}(t))$ *over all* $\mathbf{x}$ *inside the mesh is independent of* $\mathbf{s}(t)$.

However, the converse of this last implication is not true.

Proof. (a)$\Rightarrow$(b). On an arbitrary element, (2.6) says

$$\boldsymbol{\xi} = \sum_{k=0}^{n} \mathbf{s}_{e(k)}(0)\psi_{e(k)}(\boldsymbol{\xi}, \mathbf{s}(0)) \tag{3.1}$$

and (2.13) says

$$\psi_{e(k)}(\boldsymbol{\xi}(\mathbf{x}, \mathbf{s}(t)), \mathbf{s}(0)) \equiv \psi_{e(k)}(\mathbf{x}, \mathbf{s}(t)), \qquad k = 0, \cdots, n. \tag{3.2}$$

Hence,

$$\boldsymbol{\xi} = \sum_{k=0}^{n} \mathbf{s}_{e(k)}(0)\psi_{e(k)}(\mathbf{x}, \mathbf{s}(t)), \tag{3.3}$$

and from (a) we deduce that $\alpha_j(\mathbf{x}, \mathbf{s}(t))$ depends only on the linear hat functions $\psi_1(\mathbf{x}, \mathbf{s}(t)), \cdots, \psi_N(\mathbf{x}, \mathbf{s}(t))$.

(b)$\Rightarrow$(c). Again use (2.13), this time to deduce that

$$\psi_k(\boldsymbol{\xi}(\mathbf{x}, \mathbf{s}(t)), \mathbf{s}(0)) \equiv \psi_k(\mathbf{x}, \mathbf{s}(t)), \qquad k = 1, \cdots, N. \tag{3.4}$$

Now, (b)$\Rightarrow$

$$\alpha_j(\mathbf{x}, \mathbf{s}(t)) \equiv f_j(\psi_1(\mathbf{x}, \mathbf{s}(t)), \cdots, \psi_N(\mathbf{x}, \mathbf{s}(t))),$$

and, using (3.4), this becomes

$$\alpha_j(\mathbf{x}, \mathbf{s}(t)) \equiv f_j(\psi_1(\boldsymbol{\xi}, \mathbf{s}(0)), \cdots, \psi_N(\boldsymbol{\xi}, \mathbf{s}(0))). \tag{3.5}$$

Using (b) once more gives the required expression

$$\alpha_j(\mathbf{x}, \mathbf{s}(t)) \equiv \alpha_j(\boldsymbol{\xi}, \mathbf{s}(0)).$$

(c)$\Rightarrow$(a). Simply define $\bar{\alpha}_j(\boldsymbol{\xi}) \equiv \alpha_j(\boldsymbol{\xi}, \mathbf{s}(0))$.

(c)$\Rightarrow$(d). Note that the range of $\alpha_j(\boldsymbol{\xi}, \mathbf{s}(0))$, over all $\boldsymbol{\xi}$ inside the mesh when $t = 0$, is independent of $\mathbf{s}(t)$. Hence, when (c) holds, this must also be true for $\alpha_j(\mathbf{x}, \mathbf{s}(t))$.

It only remains to show that (d) can be true for some $\alpha_j(\mathbf{x}, \mathbf{s}(t))$ which does not satisfy (a), (b), or (c). The following example will suffice. In one spatial dimension, define $\alpha_j(x, \mathbf{s}(t))$ by

$$\alpha_j(x, \mathbf{s}(t)) = \begin{cases} 0 & \text{when } \psi_j(x, \mathbf{s}(t)) = 0, \\ 0 & \text{when } x = s_j(t), \\ -1 & \text{when } x = [s_{j-1} + (s_j - s_{j-1})(1 - \exp(s_{j-1} - s_j))](t), \\ 1 & \text{when } x = [s_j + (s_{j+1} - s_j)(1 - \exp(s_j - s_{j+1}))](t), \end{cases}$$

where $\alpha_j(x, \mathbf{s}(t))$ consists of four linear pieces on $[s_{j-1}(t), s_{j+1}(t)]$, and the function $\exp(\cdot)$ is the usual exponentiation.

This basis function satisfies (d) with range $[-1, 1]$, but fails to satisfy (b), for example. □

We will now show that, in any number of space dimensions n, condition (b) of the lemma above is satisfied by all Lagrange finite-element basis functions. Hence, condition (2.28) of Theorem 2.4 can be satisfied, and the theorem applied.

We must first give a definition of a Lagrange basis functions, and in order to do this we need the following result due to Nicolaides [13].

THEOREM 3.2 (Nicolaides). *Let $\mathcal{S}$ be an n-dimensional simplex with vertices at* $\mathbf{s}_0, \cdots, \mathbf{s}_n$. *Then, for a given integer $q \geqq 1$, any polynomial $p \in P_q$, the space of polynomials of degree q is uniquely determined by its values on the set*

$$L_q(\mathcal{S}) = \left\{ \mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \sum_{d=0}^{n} \lambda_d \mathbf{s}_d, \sum_{d=0}^{n} \lambda_d = 1, \lambda_d \in \left\{ 0, \frac{1}{q}, \cdots, \frac{q-1}{q}, 1 \right\}, 0 \leqq d \leqq n \right\}. \tag{3.6}$$

We are now able to give the following definition and theorem.

DEFINITION. On a simplex, $\mathcal{S}$ say, in $\mathbb{R}^n$, the Lagrange basis for the space P_q is the set of functions obtained as follows. For each point in the set $L_q(\mathcal{S})$, given by (3.6), uniquely define a corresponding function in P_q which takes the value 1 at this point and zero at all the other points.

THEOREM 3.3. *Each Lagrange basis function of degree q on a simplex $\mathcal{S}$ in $\mathbb{R}^n$ can be written as a function of $\{\psi_0(\mathbf{x}, \mathbf{s}), \cdots, \psi_n(\mathbf{x}, \mathbf{s})\}$, the set of linear hat functions on the simplex, alone.*

Proof. We proceed by induction on q.

$q = 1$. Each of the Lagrange basis functions is by definition one of the $\psi_k(\mathbf{x}, \mathbf{s})$'s, and so the result is trivial.

Now assume the result is true for P_{q-1}. Consider an arbitrary Lagrange basis function—α, say—of degree q, on our simplex $\mathcal{S}$. The simplex has $(n+1), (n-1)$-dimensional faces which are themselves simplexes numbered $0, \cdots, n$ (with face k having all of the vertices of $\mathcal{S}$ except vertex k), say.

On at least one of these $(n-1)$-dimensional faces, $\alpha(\mathbf{x}, \mathbf{s}) \equiv 0$. Suppose face j is one such face; then $\psi_j(\mathbf{x}, \mathbf{s}) \equiv 0$ on this face also. In fact, the only places where $\psi_j(\mathbf{x}, \mathbf{s}) = 0$ are the points in the hyperplane in which face j lies, hence ψ_j must be a factor of α. This means that there exists a $(q-1)$th degree polynomial—$\alpha_j(\mathbf{x}, \mathbf{s})$, say—such that

$$\alpha(\mathbf{x}, \mathbf{s}) \equiv \psi_j(\mathbf{x}, \mathbf{s}) \alpha_j(\mathbf{x}, \mathbf{s}). \tag{3.7}$$

Now define another n-dimensional simplex $\mathcal{S}_j$ as having vertices at

$$\mathbf{s}_j + \frac{(q-1)}{q}(\mathbf{s}_k - \mathbf{s}_j), \qquad k = 0, \cdots, n. \tag{3.8}$$

Then, to within a constant (which depends only on $\psi_j(\mathbf{x}, \mathbf{s})$), $\alpha_j(\mathbf{x}, \mathbf{s})$ is a Lagrange basis function of degree $q-1$ on $\mathcal{S}_j$. So, by the inductive hypothesis, $\alpha_j(\mathbf{x}, \mathbf{s})$ depends only on the Lagrange basis functions of degree 1 on $\mathcal{S}_j$. However, these are just linear functions on $\mathcal{S}$ and therefore depend only on $\{\psi_0(\mathbf{x}, \mathbf{s}), \cdots, \psi_n(\mathbf{x}, \mathbf{s})\}$. Thus $\alpha(\mathbf{x}, \mathbf{s})$ depends only on $\{\psi_0(\mathbf{x}, \mathbf{s}), \cdots, \psi_n(\mathbf{x}, \mathbf{s})\}$. $\square$

Although the result above is only given for a Lagrange basis function of degree q on a single simplex, it is easily generalized to a whole grid of simplexes. This is done by first extending the definition of a Lagrange basis function to cover the entire grid. We set

$$L_q = \cup\, L_q(\mathcal{S}) \quad \text{over all simplexes } \mathcal{S} \text{ in the grid,}$$

and then define a basis for $P_q^0(\mathbf{s})$, the space of C^0 piecewise polynomials of degree q on the grid $\mathbf{s}$, by introducing a function corresponding to each point in L_q, in the same way as we did before. Now, since the result of Theorem 3.3 is given trivially for a function which is identically zero on a simplex, the theorem must still hold for our extended Lagrange basis of $P_q^0(\mathbf{s})$.

Hence, we have shown that Theorem 2.4 may always be applied to a Lagrange finite-element basis function of degree q in n dimensions. Thus, for any function

$v \in P_q^0(\mathbf{s})$,

$$\frac{\partial v}{\partial \mathbf{s}_j}(\mathbf{x}, \mathbf{s}(t)) = -\psi_j(\mathbf{x}, \mathbf{s}(t)) \frac{\partial v}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{s}(t)). \tag{3.9}$$

4. Hermite finite elements. We will follow the notation of Ciarlet [3] and define a Hermite finite-element space to be any finite-element space such that at least one directional derivative occurs as a degree of freedom. As has already been mentioned, (2.28) cannot be satisfied for all of the basis functions of an arbitrary space of this type. The simplest example of this is given by Hermite piecewise cubic polynomials in one dimension. The conventional basis for this space is constructed as follows.

Choose a grid $\mathbf{s} \in \mathbb{R}^N$ (such that $s_1 < s_2 < \cdots < s_{N-1} < s_N$). Then, for each node, s_i, define two piecewise cubic basis functions, $\kappa_i(x, \mathbf{s})$ and $\lambda_i(x, \mathbf{s})$ say, by

$$\kappa_i(s_j, \mathbf{s}) = \delta_{ij}, \quad \frac{\partial \kappa_i}{\partial x}(s_j, \mathbf{s}) = 0, \quad j = 1, \cdots, N,$$

$$\lambda_i(s_j, \mathbf{s}) = 0, \quad \frac{\partial \lambda_i}{\partial x}(s_j, \mathbf{s}) = \delta_{ij}, \quad j = 1, \cdots, N.$$

It can easily be shown that

$$\kappa_i(x, \mathbf{s}) \equiv \psi_i^2(x, \mathbf{s})[3 - 2\psi_i(x, \mathbf{s})] \tag{4.1}$$

and

$$\lambda_i(x, \mathbf{s}) \equiv \psi_i^2(x, \mathbf{s})[x - s_i], \tag{4.2}$$

where, as usual, $\psi_i(x, \mathbf{s})$ is the linear hat function which takes the value 1 at s_i. By Lemma 3.1 and Theorem 2.4,

$$\frac{\partial \kappa_i}{\partial s_j}(x, \mathbf{s}) = -\psi_j(x, \mathbf{s}) \frac{\partial \kappa_i}{\partial x}(x, \mathbf{s}); \tag{4.3}$$

however, the analogous result does not hold for λ_i. This is because (2.28) cannot be satisfied for these functions. Observe that the range of values taken by $\lambda_i(x, \mathbf{s})$ is $[-4(s_i - s_{i-1})/27, 4(s_{i+1} - s_i)/27]$, which is clearly grid dependent, hence statement (d) of Lemma 3.1 is not satisfied, and so neither is statement (a).

Despite this difficulty we may still obtain an expression of the desired form for the grid derivatives of $\lambda_i(x, \mathbf{s})$, and hence of v. Note that

$$\frac{\partial \lambda_i}{\partial s_j}(x, \mathbf{s}) = -\psi_i^2(x, \mathbf{s})\delta_{ij} - (x - s_i)\psi_j(x, \mathbf{s}) \frac{\partial}{\partial x}(\psi_i^2(x, \mathbf{s})) \tag{4.4}$$

(using Theorem 2.4 on the $\psi_i^2(x, \mathbf{s})$ term); therefore

$$\begin{aligned} \frac{\partial \lambda_i}{\partial s_j}(x, \mathbf{s}) &= -\psi_i^2(x, \mathbf{s})\delta_{ij} - 2(x - s_i)\psi_j(x, \mathbf{s})\psi_i(x, \mathbf{s}) \frac{\partial \psi_i}{\partial x}(x, \mathbf{s}) \\ &= -\psi_j(x, \mathbf{s}) \frac{\partial \lambda_i}{\partial x}(x, \mathbf{s}) + \psi_i^2(x, \mathbf{s})[\psi_j(x, \mathbf{s}) - \delta_{ij}]. \end{aligned} \tag{4.5}$$

Hence, we may deduce that

$$\frac{\partial v}{\partial s_j}(x, \mathbf{s}) = -\psi_j(x, \mathbf{s}) \frac{\partial v}{\partial x}(x, \mathbf{s}) + \sum_{i=1}^{N} \frac{\partial v}{\partial x}(s_i, \mathbf{s})\psi_i^2(x, \mathbf{s})[\psi_j(x, \mathbf{s}) - \delta_{ij}]. \tag{4.6}$$

Thus

$$(4.7)\qquad \begin{aligned}\frac{\partial v}{\partial s_j}(s_k,\mathbf{s}) &= -\delta_{jk}\frac{\partial v}{\partial x}(s_k,\mathbf{s})+\sum_{i=1}^{N}\frac{\partial v}{\partial x}(s_i,\mathbf{s})\delta_{ik}^2(\delta_{jk}-\delta_{ij})\\ &= -\delta_{jk}\frac{\partial v}{\partial x}(s_k,\mathbf{s}),\end{aligned}$$

and

$$(4.8)\qquad \begin{aligned}\frac{\partial^2 v}{\partial x\,\partial s_j}(x_k,\mathbf{s}) = {}&-\delta_{jk}\frac{\partial^2 v}{\partial x^2}(s_k,\mathbf{s})-\frac{\partial\psi_j}{\partial x}\frac{\partial v}{\partial x}(s_k,\mathbf{s})\\ &+\sum_{i=1}^{N}\frac{\partial v}{\partial x}(s_i,\mathbf{s})\left[\psi_i^2(s_k,\mathbf{s})\frac{\partial\psi_j}{\partial x}+2\psi_i(s_k,\mathbf{s})\psi_j(s_k,\mathbf{s})\frac{\partial\psi_i}{\partial x}\right]\\ &-2\frac{\partial v}{\partial x}(s_j,\mathbf{s})\psi_j(s_k,\mathbf{s})\frac{\partial\psi_j}{\partial x}(s_k,\mathbf{s})\\ = {}&-\delta_{jk}\frac{\partial^2 v}{\partial x^2}(s_k,\mathbf{s})-\frac{\partial\psi_j}{\partial x}\frac{\partial v}{\partial x}(s_k,\mathbf{s})\\ &+\sum_{i=1}^{N}\frac{\partial v}{\partial x}(s_i,\mathbf{s})\left[\delta_{ik}\frac{\partial\psi_j}{\partial x}+2\delta_{ik}\delta_{jk}\frac{\partial\psi_i}{\partial x}\right]-2\frac{\partial v}{\partial x}(s_k,\mathbf{s})\delta_{jk}\frac{\partial\psi_j}{\partial x}\\ = {}&-\delta_{jk}\frac{\partial^2 v}{\partial x^2}(s_k,\mathbf{s}).\end{aligned}$$

Finally, (4.7) and (4.8) imply that

$$(4.9)\qquad \frac{\partial v}{\partial s_j}(x,\mathbf{s})=-\frac{\partial v}{\partial x}(s_j,\mathbf{s})\kappa_j(x,\mathbf{s})-\frac{\partial^2 v}{\partial x^2}(s_j^-,\mathbf{s})\lambda_j^-(x,\mathbf{s})-\frac{\partial^2 v}{\partial x^2}(s_j^+,\mathbf{s})\lambda_j^+(x,\mathbf{s}),$$

where $\lambda_j^-(x,\mathbf{s})$ is $\lambda_j(x,\mathbf{s})$ restricted to $[s_{j-1}, s_j]$, $\lambda_j^+(x,\mathbf{s})$ is $\lambda_j(x,\mathbf{s})$ restricted to $[s_j, s_{j+1}]$,

$$\frac{\partial^2 v}{\partial x^2}(s_j^-,\mathbf{s})=\lim_{x\uparrow s_j}\left[\frac{\partial^2 v}{\partial x^2}(x,\mathbf{s})\right],$$

and

$$\frac{\partial^2 v}{\partial x^2}(s_j^+,\mathbf{s})=\lim_{x\downarrow s_j}\left[\frac{\partial^2 v}{\partial x^2}(x,\mathbf{s})\right].$$

So despite the fact that Theorem 2.4 cannot be applied directly to the λ_j basis functions, we are still able to write $\partial v/\partial s_j(x,\mathbf{s})$ in terms of the spatial derivatives of v, which was our original goal. For this, the simplest of all Hermite cases, the same result could, with some effort, have been obtained by explicit differentiation with respect to s_j. However, for more complicated spaces, such as the one outlined below, this becomes an unrealistic alternative. Before finishing with this space however, it is worth noting that Lynch and O'Neill [7] use an incorrect expression for $\partial v/\partial s_j$ when working with Hermite cubics in one dimension. They apply relation (2.29) directly to each λ_j function and so they do not get the last two terms on the right-hand side of expression (4.9). This means that the trial solution obtained to most problems is likely to be very poor, even though the positions and values of the node points may correlate well with the true solution, as they appear to do in [7].

A commonly used trial space in two dimensions consists of piecewise cubics which are continuous at the node points of a triangular grid. Following the terminology of Strang and Fix [15] we will denote this space by Z_3. The standard basis, $\{\kappa_i, \lambda_{i,1}, \lambda_{i,2}: i = 1, \cdots, N\} \cup \{\phi_i: i = 1, \cdots, E\}$, where our finite-element grid has N nodes and E triangular elements, is defined as follows.

For each node $i = 1, \cdots, N$

$$\begin{aligned}
&\kappa_i(\mathbf{s}_j) = \delta_{ij}, && \kappa_i(\mathbf{e}_k) = 0,\\
&\frac{\partial \kappa_i}{\partial x_1}(\mathbf{s}_j) = 0, && \frac{\partial \kappa_i}{\partial x_2}(\mathbf{s}_j) = 0,\\
&\lambda_{i,1}(\mathbf{s}_j) = 0, && \lambda_{i,1}(\mathbf{e}_k) = 0,\\
&\frac{\partial \lambda_{i,1}}{\partial x_1}(\mathbf{s}_j) = \delta_{ij}, && \frac{\partial \lambda_{i,1}}{\partial x_2}(\mathbf{s}_j) = 0,\\
&\lambda_{i,2}(\mathbf{s}_j) = 0, && \lambda_{i,2}(\mathbf{e}_k) = 0,\\
&\frac{\partial \lambda_{i,2}}{\partial x_1}(\mathbf{s}_j) = 0, && \frac{\partial \lambda_{i,2}}{\partial x_2}(\mathbf{s}_j) = \delta_{ij}.
\end{aligned} \tag{4.10a}$$

For each element $i = 1, \cdots, E$

$$\begin{aligned}
&\phi_i(\mathbf{s}_j) = 0, && \phi_i(\mathbf{e}_k) = \delta_{ik},\\
&\frac{\partial \phi_i}{\partial x_1}(\mathbf{s}_j) = 0, && \frac{\partial \phi_i}{\partial x_2}(\mathbf{s}_j) = 0,
\end{aligned} \tag{4.10b}$$

where $j \in \{1, \cdots, N\}$, $k \in \{1, \cdots, E\}$, $\mathbf{s}_j$ is the position of node j, $\mathbf{e}_k$ is the position of the centroid of element k, and each basis function is a cubic polynomial on each element.

Mitchell and Wait [11] make the useful observation that on an arbitrary element—e, say—with a local numbering, $e(1)$, $e(2)$, $e(3)$, of the vertices

$$\phi_e = 27\psi_{e(1)}\psi_{e(2)}\psi_{e(3)}, \tag{4.11}$$

$$\kappa_{e(j)} = \psi_{e(j)}[3\psi_{e(j)} - 2\psi_{e(j)}^2 - 7\psi_{e(k)}\psi_{e(l)}], \tag{4.12}$$

$$\begin{aligned}
\lambda_{e(j),1} = \psi_{e(j)}[&(s_{e(j),1} - s_{e(k),1})\psi_{e(k)}(\psi_{e(l)} - \psi_{e(j)})\\
&+ (s_{e(j),1} - s_{e(l),1})\psi_{e(l)}(\psi_{e(k)} - \psi_{e(j)})],
\end{aligned} \tag{4.13a}$$

and

$$\begin{aligned}
\lambda_{e(j),2} = \psi_{e(j)}[&(s_{e(j),2} - s_{e(k),2})\psi_{e(k)}(\psi_{e(l)} - \psi_{e(j)})\\
&+ (s_{e(j),2} - s_{e(l),2})\psi_{e(l)}(\psi_{e(k)} - \psi_{e(j)})],
\end{aligned} \tag{4.13b}$$

where (j, k, l) is any cyclic permutation of $(1, 2, 3)$. Note that on each element with node j as a vertex (and other vertices at $\mathbf{s}_k$ and $\mathbf{s}_l$, say)

$$\psi_j(\mathbf{x}, \mathbf{s}) = \frac{(s_{k1}s_{l2} - s_{k2}s_{l1}) + (s_{k2} - s_{l2})x_1 - (s_{k1} - s_{l1})x_2}{(s_{k1}s_{l2} - s_{k2}s_{l1}) - (s_{j1}s_{l2} - s_{j2}s_{l1}) + (s_{j1}s_{k2} - s_{j2}s_{k1})},$$

so it is easy to see the difficulty we would have in differentiating (4.12) and (4.13), for example, with respect to $\mathbf{s}_i$ if it was done explicitly. However, by Lemma 3.1 and Theorem 2.4, we know that on this element

$$\frac{\partial \phi_e}{\partial \mathbf{s}_{e(i)}}(\mathbf{x}, \mathbf{s}) = -\psi_{e(i)}(\mathbf{x}, \mathbf{s})\frac{\partial \phi_e}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{s}) \tag{4.14}$$

and

$$\frac{\partial \kappa_{e(j)}}{\partial \mathbf{s}_{e(i)}}(\mathbf{x}, \mathbf{s}) = -\psi_{e(i)}(\mathbf{x}, \mathbf{s}) \frac{\partial \kappa_{e(j)}}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{s}). \tag{4.15}$$

Lemma 3.1 also tells us that we cannot use Theorem 2.4 directly on $\lambda_{e(j),f}$ $(f = 1, 2)$, but that we can apply it to $\psi_{e(j)}\psi_{e(k)}(\psi_{e(l)} - \psi_{e(j)})$ and $\psi_{e(j)}\psi_{e(l)}(\psi_{e(k)} - \psi_{e(j)})$. For $d = 1, 2$ this gives

$$\begin{aligned} \frac{\partial \lambda_{e(j),f}}{\partial s_{e(i),d}} &= [\psi_{e(j)}\psi_{e(k)}(\psi_{e(l)} - \psi_{e(j)})] \frac{\partial}{\partial s_{e(i),d}}(s_{e(j),f} - s_{e(k),f}) \\ &\quad - (s_{e(j),f} - s_{e(k),f})\psi_{e(i)} \frac{\partial}{\partial x_d}[\psi_{e(j)}\psi_{e(k)}(\psi_{e(l)} - \psi_{e(j)})] \\ &\quad + [\psi_{e(j)}\psi_{e(l)}(\psi_{e(k)} - \psi_{e(j)})] \frac{\partial}{\partial s_{e(i),d}}(s_{e(j),f} - s_{e(l),f}) \\ &\quad - (s_{e(j),f} - s_{e(l),f})\psi_{e(i)} \frac{\partial}{\partial x_d}[\psi_{e(j)}\psi_{e(l)}(\psi_{e(k)} - \psi_{e(j)})] \\ &= -\psi_{e(i)} \frac{\partial \lambda_{e(j),f}}{\partial x_d} + \delta_{fd}[\psi_{e(j)}\psi_{e(k)}(\psi_{e(l)} - \psi_{e(j)})(\delta_{ij} - \delta_{ik}) \\ &\quad + \psi_{e(j)}\psi_{e(l)}(\psi_{e(k)} - \psi_{e(j)})(\delta_{ij} - \delta_{il})]. \end{aligned} \tag{4.16}$$

Hence, combining (4.14)-(4.16), we get

$$\begin{aligned} \frac{\partial v}{\partial s_{e(i),d}} = -\psi_{e(i)} \frac{\partial v}{\partial x_d} + \sum_{j=1}^{3} \bigg\{ \frac{\partial v}{\partial x_d}(\mathbf{s}_{e(j)}, \mathbf{s})[\psi_{e(j)}\psi_{e(k)}(\psi_{e(l)} - \psi_{e(j)})(\delta_{ij} - \delta_{ik}) \\ + \psi_{e(j)}\psi_{e(l)}(\psi_{e(k)} - \psi_{e(j)})(\delta_{ij} - \delta_{il})] \bigg\}, \end{aligned} \tag{4.17}$$

where (j, k, l) is still any cyclic permutation of $(1, 2, 3)$. So we now have an expression for $\partial v/\partial s_{e(i),d}$, however the right-hand side of (4.17) may still be expressed in a more convenient form. Let (p, q, r) also be any cyclic permutation of $(1, 2, 3)$. We will evaluate $\partial v/\partial s_{e(i),d}(\mathbf{s}_{e(p)}, \mathbf{s})$, $\partial v/\partial s_{e(i),d}(\mathbf{e}, \mathbf{s})$, and $\partial^2 v/\partial x_f\, \partial s_{e(i),d}(\mathbf{s}_{e(p)}, \mathbf{s})$:

$$\frac{\partial v}{\partial s_{e(i),d}}(\mathbf{s}_{e(p)}, \mathbf{s}) = -\delta_{ip} \frac{\partial v}{\partial x_d}(\mathbf{s}_{e(p)}, \mathbf{s}), \tag{4.18}$$

$$\frac{\partial v}{\partial s_{e(i),d}}(\mathbf{e}, \mathbf{s}) = -\frac{1}{3} \frac{\partial v}{\partial x_d}(\mathbf{e}, \mathbf{s}), \tag{4.19}$$

and

$$\begin{aligned} &\frac{\partial^2 v}{\partial x_f\, \partial s_{e(i),d}}(\mathbf{s}_{e(p)}, \mathbf{s}) \\ &= -\delta_{ip} \frac{\partial^2 v}{\partial x_f\, \partial x_d}(\mathbf{s}_{e(p)}, \mathbf{s}) - \frac{\partial \psi_{e(i)}}{\partial x_f} \frac{\partial v}{\partial x_d}(\mathbf{s}_{e(p)}, \mathbf{s}) \\ &\quad + \sum_{j=1}^{3} \bigg\{ \frac{\partial v}{\partial x_d}(\mathbf{s}_{e(j)}, \mathbf{s}) \bigg[(\delta_{lp} - \delta_{jp}) \bigg(\delta_{jp} \frac{\partial \psi_{e(k)}}{\partial x_f} + \delta_{kp} \frac{\partial \psi_{e(j)}}{\partial x_f} \bigg)(\delta_{ij} - \delta_{ik}) \\ &\quad + (\delta_{kp} - \delta_{jp}) \bigg(\delta_{jp} \frac{\partial \psi_{e(l)}}{\partial x_f} + \delta_{lp} \frac{\partial \psi_{e(j)}}{\partial x_f} \bigg)(\delta_{ij} - \delta_{il}) \bigg] \bigg\}, \end{aligned}$$

$$(4.20) \qquad = -\delta_{ip}\frac{\partial^2 v}{\partial x_f\,\partial x_d}(\mathbf{s}_{e(p)},\mathbf{s}) - \frac{\partial \psi_{e(i)}}{\partial x_f}\frac{\partial v}{\partial x_d}(\mathbf{s}_{e(p)},\mathbf{s})$$

$$+(0-1)\left(\frac{\partial \psi_{e(q)}}{\partial x_f}\right)(\delta_{ip}-\delta_{iq})\frac{\partial v}{\partial x_d}(\mathbf{s}_{e(p)},\mathbf{s})$$

$$+(0-1)\left(\frac{\partial \psi_{e(r)}}{\partial x_f}\right)(\delta_{ip}-\delta_{ir})\frac{\partial v}{\partial x_d}(\mathbf{s}_{e(p)},\mathbf{s}),$$

$$(4.21) \qquad = -\delta_{ip}\frac{\partial^2 v}{\partial x_f\,\partial x_d}(\mathbf{s}_{e(p)},\mathbf{s}).$$

Using (4.18), (4.19), and (4.21) we can deduce that, on an arbitrary element,

$$(4.22) \qquad \begin{aligned} \frac{\partial v}{\partial s_{e(i)d}} = -\frac{\partial v}{\partial x_d}(\mathbf{s}_{e(i)},\mathbf{s})\kappa_{e(i)} - \frac{1}{3}\frac{\partial v}{\partial x_d}(\mathbf{e},\mathbf{s})\phi_e \\ -\frac{\partial^2 v}{\partial x_1\,\partial x_d}(\mathbf{s}_{e(i)},\mathbf{s})\lambda_{e(i),1} - \frac{\partial^2 v}{\partial x_2\,\partial x_d}(\mathbf{s}_{e(i)},\mathbf{s})\lambda_{e(i),2}. \end{aligned}$$

Once more, we have been able to write the derivatives with respect to node movement in terms of conventional spatial derivatives. Most finite-element algorithms involve working inside a loop over all of the elements, and so (4.22) gives a convenient form of the result. However, it may easily be written in global notation as

$$\frac{\partial v}{\partial \mathbf{s}_j} = -\frac{\partial v}{\partial \mathbf{x}}(\mathbf{s}_j,\mathbf{s})\kappa_j - \sum_{k=1}^{N(j)}\left[(\lambda_{j,1}^{e(j,k)},\lambda_{j,2}^{e(j,k)})^T\cdot\left(\nabla\frac{\partial v}{\partial \mathbf{x}}\right)(\mathbf{s}_j^{e(j,k)},\mathbf{s}) + \frac{1}{3}\frac{\partial v}{\partial x_d}(\mathbf{e}(j,k),\mathbf{s})\phi_{e(j,k)}\right],$$

where

$N(j)$ = the number of elements with node j as a vertex.

$e(j,k)$ = the element number of the kth element around node j, according to some fixed ordering.

$\mathbf{e}(j,k)$ = the centroid of element $e(j,k)$.

$\lambda_{j,d}^e = \lambda_{j,d}$ restricted to element e.

$(\nabla(\partial v/\partial \mathbf{x}))(\mathbf{s}_j^e,\mathbf{s})$ = the limit of $(\nabla(\partial v/\partial \mathbf{x}))(\mathbf{x},\mathbf{s})$ as $\mathbf{x}$ approaches $\mathbf{s}_j$ along any path in element e.

5. Discussion. In this paper we have observed that for any finite-element method of directly solving (1.1) on a continuously adapting grid it will be necessary to evaluate $\partial v/\partial \mathbf{s}_i$, $i=1,\cdots,N$, where v is the trial solution. Theorem 2.4 allows us to do this on a grid of simplexes, provided each basis function satisfies condition (2.28), giving an expression which depends on $\partial v/\partial \mathbf{x}$. This is a very useful result because the spatial derivatives of v need to be calculated anyway in order to evaluate $\mathscr{L}(v)$.

In § 3 we showed two equivalent statements to (2.28) and proved that Theorem 2.4 holds for Lagrange finite-element spaces of any degree and spatial dimension, thus giving a simple way of calculating otherwise unmanageable derivatives. Even when (2.28) is not satisfied by all of the basis functions of a space, the theorem can still be of great use in evaluating $\partial v/\partial \mathbf{s}_i$ as is indicated in the case of Hermite finite-element spaces in § 4.

Finally, note that although the results of this paper necessarily apply only to finite-element spaces defined on grids of simplexes, it is possible to extend them for some constrained motion of other grids. For example, if a grid moves such that all of the elements are always rectangular, then any conventional finite-element basis is such that each function can be written as a product of separate functions of each coordinate direction (e.g., bilinear and biquadratic in two dimensions or trilinear and triquadratic in three dimensions). Hence the above results may be applied to each of these functions of a single variable.

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