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On steady and large time solutions of the semi-discrete Moving Finite Element equations for one-dimensional diffusion problems

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This paper considers how the Moving Finite Element (MFE) method approximates the steady and large time solutions of a family of linear diffusion equations in one space dimension. In particular, it is shown that any steady solution to the Moving Finite Element equations must satisfy the stationary equations for a best approximation to the steady solution of the PDE from the manifold of free-knot linear splines, in some problem dependent norm.

For the special case of the inhomogeneous linear heat equation it is also shown that, under certain conditions, the only steady MFE solution is the unique global best fit to the true steady solution, in the H^1 semi-norm. It is also demonstrated numerically that these steady solutions are stable attractors. Finally, a numerical study of the large time solutions of the homogeneous linear heat equation is undertaken and it is demonstrated that the MFE solutions appear to possess a rather novel temporal accuracy property.

1. Introduction

The Moving Finite Element method for the solution of time-dependent partial differential equations was first introduced by Miller *et al.* ([20], [21], and [8]) in 1981. It is a finite element method in which a spatial mesh with a constant number of degrees of freedom is allowed to deform continuously in time. Unlike in [9], [18] or [24] for example, this is achieved without tying the node positions to individually tracked solution properties such as characteristic speeds or the motions of internal boundaries. Instead these positions are treated as unknown time-dependent variables which, just like the conventional finite element degrees of freedom, must be evaluated as part of the solution procedure. This procedure is designed to determine simultaneously at each time both the best (in some sense which is defined below) possible spatial mesh and the best approximation to the solution on that mesh.

In this section we give a brief description of the Moving Finite Element method in 1-*d* so as to introduce the notation of the rest of the paper. We do not give any unnecessary details of the procedure or its implementation since these are clearly outlined in a number of papers such as [8], [2] or [26]. We also introduce the partial differential equations which we are concerned with solving, which belong to a class of linear diffusion problems.

In Section 2 we derive the main result of the paper. This provides a relationship between the steady solutions of the Moving Finite Element semi-discretization of the PDE and the locally best linear spline approximations to the steady solution of the PDE, in some problem dependent norm. After a discussion of some of the consequences, and limitations, of this result further analysis is performed in Section 3. Here it is necessary to impose more restrictive conditions on the type of equation considered but we are able to show that the steady MFE solution may be unique, and equal to the globally best approximation of the true steady solution. Finally, Section 4 presents some numerical investigations which complement and extend the theoretical results.

As has been mentioned above we start by giving a brief outline of the Moving Finite Element method. For the purposes of this paper we will be concerned with an evolution equation of the form

$$\frac{\partial u}{\partial t}(x, t) = \frac{\partial}{\partial x}\left[p(x)\frac{\partial u}{\partial x}(x, t)\right] - q(x)u(x, t) + r(x), \qquad 0 \leq x \leq 1, 0 \leq t, \quad (1.1)$$

where,

$$p \in C^1[0, 1], \quad p > 0$$
$$q, r \in C^0[0, 1], \quad q \geq 0,$$

and, for the simplicity of this introduction, we will only impose homogeneous Dirichlet boundary conditions at $x = 0$ and $x = 1$. We will also assume that the initial data satisfies these boundary conditions. We will now seek an approximation, v, to u, on a time-dependent grid denoted by $s(t)$, of the form

$$v(x, t) = \sum_{i=1}^{N} a_i(t)\alpha_i(x, s(t)), \quad (1.2)$$

where $s(t) \in T = \{s \in \Re^N : s_0 = 0 < s_1 < \cdots < s_N < 1 = s_{N+1}\}$, and each $\alpha_i(x, s(t))$ is the unique linear spline on the grid $s(t)$ satisfying

$$\alpha_i(s_j, s(t)) = \delta_{ij}$$

for $i \in \{1, \ldots, N\}$ and $j \in \{0, \ldots, N+1\}$. In order to obtain this approximation the Moving Finite Element method attempts to solve a weak form of (1.1), for which the test space at any given time is the space in which $(\partial v/\partial t)$ lies. Differentiating (1.2) with respect to time gives

$$\frac{\partial v}{\partial t} = \sum_{i=1}^{N}\left(\dot{a}_i\alpha_i + a_i\sum_{j=1}^{N}\left(\frac{\partial \alpha_i}{\partial s_j}\dot{s}_j\right)\right) = \sum_{i=1}^{N}\left(\dot{a}_i\alpha_i + \dot{s}_i\frac{\partial v}{\partial s_i}\right) = \sum_{i=1}^{N}(\dot{a}_i\alpha_i + \dot{s}_i\beta_i),$$

where $\beta_i(x, s(t)) = (\partial v/\partial s_i) = -\alpha_i(\partial v/\partial x)$ (see [19] or [11] for details). Hence one possible weak formulation of (1.1) which would ensure that at any time the residual of the PDE is orthogonal to $\partial v/\partial t$ is given formally by

$$\left\langle \sum_{i=1}^{N}(\dot{a}_i\alpha_i + \dot{s}_i\beta_i), \alpha_j \right\rangle = \left\langle \frac{\partial}{\partial x}\left[p\frac{\partial v}{\partial x}\right], \alpha_j \right\rangle - \langle qv - r, \alpha_j \rangle \quad (1.3)$$

and

$$\left\langle \sum_{i=1}^{N}(\dot{a}_i\alpha_i + \dot{s}_i\beta_i), \beta_j \right\rangle = \left\langle \frac{\partial}{\partial x}\left[p\frac{\partial v}{\partial x}\right], \beta_j \right\rangle - \langle qv - r, \beta_j \rangle \quad (1.4)$$

for $j=1, \dots, N$, where $\langle \cdot, \cdot \rangle$ represents the usual $L^2[0, 1]$ inner product. Unfortunately however the second of these sets of equations is not properly defined since the term $\langle (\partial/\partial x)[p(\partial v/\partial x)], -(\partial v/\partial x)\alpha_j \rangle$ does not exist, even in a distributional sense. However, by using the following formal integration by parts argument of Mueller [22] it is possible to overcome this difficulty:

$$\begin{aligned}
\left\langle \frac{\partial}{\partial x}\left[p\frac{\partial v}{\partial x}\right], -\frac{\partial v}{\partial x}\alpha_j \right\rangle &= -\left\langle p\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}, \alpha_j \right\rangle - \left\langle \frac{\mathrm{d}p}{\mathrm{d}x}\left[\frac{\partial v}{\partial x}\right]^2, \alpha_j \right\rangle \\
&= -\frac{1}{2}\left\langle \frac{\partial}{\partial x}\left\{\left[\frac{\partial v}{\partial x}\right]^2\right\}, p\alpha_j \right\rangle - \left\langle \frac{\mathrm{d}p}{\mathrm{d}x}\left[\frac{\partial v}{\partial x}\right]^2, \alpha_j \right\rangle \\
&= \frac{1}{2}\left\langle \left[\frac{\partial v}{\partial x}\right]^2, p\frac{\partial \alpha_j}{\partial x} + \frac{\mathrm{d}p}{\mathrm{d}x}\alpha_j \right\rangle - \left\langle \frac{\mathrm{d}p}{\mathrm{d}x}\left[\frac{\partial v}{\partial x}\right]^2, \alpha_j \right\rangle \\
&= \frac{1}{2}\left\langle p\left[\frac{\partial v}{\partial x}\right]^2, \frac{\partial \alpha_j}{\partial x} \right\rangle - \frac{1}{2}\left\langle \frac{\mathrm{d}p}{\mathrm{d}x}\left[\frac{\partial v}{\partial x}\right]^2, \alpha_j \right\rangle.
\end{aligned}$$

This last expression is well-defined, and if we use it in our definition of the Moving Finite Element method (in equation (1.4)) it is equivalent to making a certain smoothing of the functions in our test space (see [12] and [6] for a full description of this and alternative ways of treating terms with second derivatives). Equations (1.3) and (1.4) then become

$$\sum_{i=1}^{N} \langle \alpha_j, \alpha_i \rangle \dot{a}_i + \sum_{i=1}^{N} \langle \alpha_j, \beta_i \rangle \dot{s}_i = -\left\langle p\frac{\partial v}{\partial x}, \frac{\partial \alpha_j}{\partial x} \right\rangle - \langle qv - r, \alpha_j \rangle \tag{1.5}$$

and

$$\begin{aligned}
\sum_{i=1}^{N} \langle \beta_j, \alpha_i \rangle \dot{a}_i + \sum_{i=1}^{N} \langle \beta_j, \beta_i \rangle \dot{s}_i = \frac{1}{2}\left\langle p\left[\frac{\partial v}{\partial x}\right]^2, \frac{\partial \alpha_j}{\partial x} \right\rangle - \frac{1}{2}\left\langle \frac{\mathrm{d}p}{\mathrm{d}x}\left[\frac{\partial v}{\partial x}\right]^2, \alpha_j \right\rangle \\
- \langle qv - r, \beta_j \rangle
\end{aligned} \tag{1.6}$$

for $j=1, \dots, N$. We will refer to these as the Moving Finite Element equations and may write them in the form

$$A(y)\dot{y} = g(y), \tag{1.7}$$

where

$$\begin{aligned}
y &= (a_1, s_1, \dots, a_N, s_N)^T, \\
\boldsymbol{\alpha} &= (\alpha_1, \beta_1, \dots, \alpha_N, \beta_N)^T, \\
A &= \langle \boldsymbol{\alpha}, \boldsymbol{\alpha}^T \rangle
\end{aligned}$$

and g is the vector of right-hand-sides. The matrix $A(y)$ will be referred to as the 'MFE mass matrix' by analogy with the usual Galerkin mass matrix.

It should be noted that even though (1.1) is a linear PDE, the Moving Finite Element semi-discretization yields a non-linear system of ordinary differential equations. Also the matrix A, although positive semi-definite, may sometimes become singular. This occurs when the elements of the ordered set $\boldsymbol{\alpha}$, defined above, form a linearly dependent set. This is equivalent to $\partial v/\partial x$ being continuous at one or more nodes which, since v is only a piecewise linear function, corresponds to the slope of the MFE solution being the same across two neighbouring elements. Hence such a singularity in A will be said to be caused by 'parallelism' and the MFE problem will then be said to be 'degenerate'.

The problem of parallelism along with the possibility of two or more nodes colliding with each other appear to be the two major drawbacks of the MFE method. One approach to overcoming these difficulties is to attempt to influence the nodal motion by using penalty functions in the underlying minimization to which equations (1.5) and (1.6) correspond. This is the approach of Miller *et al.* ([20], [21], [8]) and Mueller & Carey [23] for example. However, the work of Baines *et al.* ([2], [27], [3], [4], [12]), mainly, but not exclusively, for hyperbolic PDE's, suggests that the use of these awkward-to-handle penalty functions may not always be necessary. Computational experience of the author ([10]) also suggests that this is the case for certain problems, such as those being considered here.

The fact that the MFE method appears to work in practice is one of the main reasons for analyzing it on this particular class of problems. The hope is to gain sufficient insight into the method to be able to predict when and why it will be efficient and, equally importantly, when and why it will not. There are other advantages to restricting ourselves to these PDEs as well. For example, it is known that problem (1.1) always has a unique steady solution (even when solved with more general boundary conditions than those so far considered) and so we know how to expect the solutions to behave for large times. Another argument in favour of restricting our analysis to these particular problems is the difficulty of doing anything more general. The only serious analysis that has been made so far is that of Dupont [7], who proves that for a certain class of parabolic equations with smooth solutions the Moving Finite Element method, under the influence of sufficiently strong penalty functions of the type used by Miller [21], is asymptotically no worse than a fixed-grid method. We will improve this statement by showing that, under a number of conditions, the MFE method in the absence of any penalty terms is, in some sense, at least as good as the best possible fixed-grid method.

2. Steady solutions of the Moving Finite Element equations

An important property of equation (1.1) is that, with Dirichlet boundary conditions at $x = 0$ and $x = 1$, it always possesses a unique steady solution (see [16] for example). Moreover, since the equation is linear it is not difficult to show that this steady solution must also be a stable attractor. This result is actually true for a wider class of boundary conditions than just Dirichlet ones (as illustrated in chapter one of [15]) but we will not explicitly consider other such cases here. In fact throughout this section it will be assumed, for simplicity, that the Dirichlet conditions to be imposed are homogeneous, with the extension to non-homogeneous conditions being quite straightforward.

Since the solution of the PDE that we are attempting to solve tends to a steady state as $t \to \infty$ it is to be hoped that the solutions of MFE equations (1.7), which are a semi-discretization of (1.1), also tend to a corresponding steady state as $t \to \infty$. It is with this in mind that we now investigate the elliptic form of equation (1.7) which is obtained by setting $\dot{\mathbf{y}}(t) = \mathbf{0}$.

If there exists a steady solution of the Moving Finite Element equations, $\mathbf{y}$, then this solution must satisfy $\mathbf{g}(\mathbf{y}) = \mathbf{0}$ in (1.7). Moreover whenever $\mathbf{g}(\mathbf{y}) = \mathbf{0}$ and the MFE mass matrix, $A(\mathbf{y})$, is non-singular, $\mathbf{y}$ must be a steady solution. Hence we will now look for solutions of the algebraic equations $\mathbf{g}(\mathbf{y}) = \mathbf{0}$. In the following theorem it is shown that these equations are in fact exactly the same equations that are satisfied by a locally best approximation to the unique steady solution, $f(x)$ say, of the PDE (1.1) in the problem dependent norm, $\|\cdot\|$, given by

$$\|u\| = \left\{ \int_0^1 \left(p\left[\frac{\partial u}{\partial x}\right]^2 + qu^2 \right) \mathrm{d}x \right\}^{\frac{1}{2}}. \tag{2.1}$$

THEOREM 2.1 Let $f(x)$ be the unique strong steady solution of (1.1);

$$\frac{\partial u}{\partial t}(x, t) = \frac{\partial}{\partial x}\left[p(x)\frac{\partial u}{\partial x}(x, t)\right] - q(x)u(x, t) + r(x), \quad 0 \leq x \leq 1, \quad 0 \leq t,$$

where,

$$p \in C^1[0, 1], \quad p > 0$$
$$q, r \in C^0[0, 1], \quad q \geq 0,$$

subject to $u(0, t) = u(1, t) = 0$. Then the following statements are equivalent:

(a) v is a steady solution of the Moving Finite Element equations with N free knots.

(b) v satisfies the stationary equations for a best linear spline approximation to f (subject to the Dirichlet boundary conditions), with N free knots, in the norm

$$\|u\| = \left\{ \int_0^1 \left(p\left[\frac{\partial u}{\partial x}\right]^2 + qu^2 \right) \mathrm{d}x \right\}^{\frac{1}{2}}.$$

Proof. We simply derive the equations that define v in each case, and show that they are the same. (a) For $j = 1, \ldots, N$, from (1.5) and (1.6), the two sets of steady MFE equations are

$$0 = -\left\langle p \sum_{i=1}^{N} a_i \frac{\partial \alpha_i}{\partial x}, \frac{\partial \alpha_j}{\partial x} \right\rangle - \langle qv - r, \alpha_j \rangle \tag{2.2}$$

and

$$0 = \frac{1}{2}\left\langle p\left[\frac{\partial v}{\partial x}\right]^2, \frac{\partial \alpha_j}{\partial x} \right\rangle - \frac{1}{2}\left\langle \frac{\mathrm{d}p}{\mathrm{d}x}\left[\frac{\partial v}{\partial x}\right]^2, \alpha_j \right\rangle - \langle qv - r, \beta_j \rangle. \tag{2.3}$$

This last set of equations can be expressed as

$$0=\int_{s_{j-1}}^{s_j} p\left[\frac{a_j-a_{j-1}}{s_j-s_{j-1}}\right]^2\frac{dx}{(s_j-s_{j-1})}-\int_{s_j}^{s_{j+1}} p\left[\frac{a_{j+1}-a_j}{s_{j+1}-s_j}\right]^2\frac{dx}{(s_{j+1}-s_j)}$$
$$-\int_{s_{j-1}}^{s_j}\frac{dp}{dx}\left[\frac{a_j-a_{j-1}}{s_j-s_{j-1}}\right]^2\left(\frac{x-s_{j-1}}{s_j-s_{j-1}}\right)dx-\int_{s_j}^{s_{j+1}}\frac{dp}{dx}\left[\frac{a_{j+1}-a_j}{s_{j+1}-s_j}\right]^2\left(\frac{s_{j+1}-x}{s_{j+1}-s_j}\right)dx$$
$$-2\langle qv-r,\beta_j\rangle$$
$$=\frac{(a_j-a_{j-1})^2}{(s_j-s_{j-1})^3}\int_{s_{j-1}}^{s_j} p\,dx-\frac{(a_{j+1}-a_j)^2}{(s_{j+1}-s_j)^3}\int_{s_j}^{s_{j+1}} p\,dx$$
$$-\frac{(a_j-a_{j-1})^2}{(s_j-s_{j-1})^3}\left\{[(x-s_{j-1})p]_{s_{j-1}}^{s_j}-\int_{s_{j-1}}^{s_j} p\,dx\right\}$$
$$-\frac{(a_{j+1}-a_j)^2}{(s_{j+1}-s_j)^3}\left\{[(s_{j+1}-x)p]_{s_j}^{s_{j+1}}+\int_{s_j}^{s_{j+1}} p\,dx\right\}-2\langle qv-r,\beta_j\rangle.$$

This, in turn, simplifies to

$$p(s_j)\frac{(a_j-a_{j-1})^2}{(s_j-s_{j-1})^2}-p(s_j)\frac{(a_{j+1}-a_j)^2}{(s_{j+1}-s_j)^2}-2\frac{(a_j-a_{j-1})^2}{(s_j-s_{j-1})^3}\int_{s_{j-1}}^{s_j} p\,dx$$
$$+2\frac{(a_{j+1}-a_j)^2}{(s_{j+1}-s_j)^3}\int_{s_j}^{s_{j+1}} p\,dx+2\left\langle qv-r,\frac{\partial v}{\partial s_j}\right\rangle=0. \quad (2.4)$$

(b) For $j=1,\ldots,N$, the two sets of stationary equations for a best approximation, v, to $f(x)$ in the given norm are

$$\frac{\partial}{\partial a_j}\int_0^1\left\{p\left[\frac{\partial v}{\partial x}-\frac{df}{dx}\right]^2+q[v-f]^2\right\}dx=0 \quad (2.5)$$

and

$$\frac{\partial}{\partial s_j}\int_0^1\left\{p\left[\frac{\partial v}{\partial x}-\frac{df}{dx}\right]^2+q[v-f]^2\right\}dx=0. \quad (2.6)$$

Using (1.2) the first of these sets of equations is simply

$$\int_0^1 p\left[\sum_{i=1}^N a_i\frac{\partial\alpha_i}{\partial x}-\frac{df}{dx}\right]\frac{\partial\alpha_j}{\partial x}dx+\int_0^1 q\left[\sum_{i=1}^N a_i\alpha_i-f\right]\alpha_j\,dx=0. \quad (2.7)$$

However, since f is a steady solution of (1.1), it follows that

$$-\int_0^1\left[p\frac{df}{dx}\frac{\partial\alpha_j}{\partial x}+qf\alpha_j\right]dx=-\int_0^1 r\alpha_j\,dx,$$

and so equations (2.7) reduce to equations (2.2) of part (a).

We now complete the proof by showing that the second set of equations above, (2.6), are exactly the same as equations (2.4). Expanding out (2.6) gives

$$\frac{\partial}{\partial s_j}\int_0^1 p\left[\frac{\partial v}{\partial x}\right]^2dx-2\frac{\partial}{\partial s_j}\int_0^1 p\frac{\partial v}{\partial x}\frac{df}{dx}dx+\frac{\partial}{\partial s_j}\int_0^1 qv^2\,dx-2\frac{\partial}{\partial s_j}\int_0^1 qvf\,dx=0,$$

i.e., using (1.2) again,

$$\frac{\partial}{\partial s_j}\int_{s_{j-1}}^{s_j} p\left[\frac{a_j - a_{j-1}}{s_j - s_{j-1}}\right]^2 \mathrm{d}x + \frac{\partial}{\partial s_j}\int_{s_j}^{s_{j+1}} p\left[\frac{a_{j+1} - a_j}{s_{j+1} - s_j}\right]^2 \mathrm{d}x$$
$$- 2\int_0^1 p \frac{\mathrm{d}f}{\mathrm{d}x}\frac{\partial}{\partial s_j}\left[\frac{\partial v}{\partial x}\right] \mathrm{d}x + 2\int_0^1 qv\frac{\partial v}{\partial s_j}\mathrm{d}x - 2\int_0^1 qf\frac{\partial v}{\partial s_j}\mathrm{d}x = 0,$$

which can be expressed as

$$\left[\frac{a_j - a_{j-1}}{s_j - s_{j-1}}\right]^2 p(s_j) - 2\frac{(a_j - a_{j-1})^2}{(s_j - s_{j-1})^3}\int_{s_{j-1}}^{s_j} p\,\mathrm{d}x - \left[\frac{a_{j+1} - a_j}{s_{j+1} - s_j}\right]^2 p(s_j)$$
$$+ 2\frac{(a_{j+1} - a_j)^2}{(s_{j+1} - s_j)^2}\int_{s_j}^{s_{j+1}} p\,\mathrm{d}x + 2\int_0^1 \frac{\partial}{\partial x}\left[p\frac{\mathrm{d}f}{\mathrm{d}x}\right]\frac{\partial v}{\partial s_j}\mathrm{d}x + 2\int_0^1 qv\frac{\partial v}{\partial s_j}\mathrm{d}x$$
$$- 2\int_0^1 qf\frac{\partial v}{\partial s_j}\mathrm{d}x = 0. \quad (2.8)$$

Now, we again use the fact that f is a steady solution of (1.1), and so

$$2\int_0^1 \left\{\frac{\partial}{\partial x}\left[p\frac{\mathrm{d}f}{\mathrm{d}x}\right] - qf\right\}\frac{\partial v}{\partial s_j}\mathrm{d}x = -2\int_0^1 r\frac{\partial v}{\partial s_j}\mathrm{d}x,$$

to deduce that equations (2.8) are precisely equations (2.4), as claimed. □

The above theorem tells us that if there are any locally best approximations to the true steady solution of the PDE in the norm (2.1), then these are also steady MFE solutions for this problem. In particular any globally best approximation to $f(x)$ from $\Re^N \times T$ (see (1.2)) will be a steady solution of the MFE equations. Should such a steady solution be a stable attractor then the Moving Finite Element method would be optimal for these equations (using the norm of Theorem 2.1).

In addition, Theorem 2.1 also allows us to use a number of results from approximation theory to learn more about steady MFE solutions. Unfortunately not all such results are necessarily positive ones. For example, it is known that there are often many solutions of the stationary equations for best spline approximations with free knots which do not correspond to globally best approximations (see [13] or [14] for the case of the L^2 norm for example). This is because the norm of the error need only be at a local minimum, or even a saddle point or a local maximum, in the parameter space, for the stationary equations to be satisfied. Moreover in [13], Jupp shows that there may even be solutions of these equations for which two or more nodes occupy the same position. Such solutions do not lie in the open set $\Re^N \times T$ of course, but rather on its boundary. However if they do exist and if the corresponding MFE solutions were to be stable attractors, then we would have the highly undesirable situation of nodes tending toward each other as $t \to \infty$.

Before attempting to investigate the consequences of Theorem 2.1 in more detail it should be noted that the result can easily be extended to cover more general boundary conditions. For example, provided there is a constant Dirichlet condition at at least one of the end points, $x = 0$ or $x = 1$, and provided that the other boundary condition is such that the PDE (1.1) has a unique solution, then the result will still hold. Also, in the case where $p(x) \equiv 0$, and so we just have an ordinary differential equation without any spatial boundary conditions, the result still goes through. In this case the norm (2.1) reduces to a weighted L^2 norm.

In the next section we will consider the MFE solution for a special case of equation (1.1); the linear heat equation. This corresponds to setting $p(x) \equiv 1$ and $q(x) \equiv 0$, although all of the work contained in the section will extend to the slightly more general case where $p \equiv$ *constant*.

3. The linear heat equation

We now restrict our attention to the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + r(x), \quad 0 \leqslant x \leqslant 1, \quad 0 \leqslant t, \tag{3.1}$$

subject to homogeneous Dirichlet boundary conditions. For this particular equation we will show that under certain conditions on the source term, $r(x)$, the MFE equations have a unique steady solution which is the unique globally best approximation to the steady solution, $f(x)$, of the PDE in the H^1 semi-norm. We are interested in this semi-norm (which is actually a genuine norm when applied to the space of functions satisfying the Dirichlet boundary conditions) because this is what is obtained from (2.1) when $p(x) \equiv 1$ and $q(x) \equiv 0$.

It will be useful to begin this section by proving two lemmas. The first of these gives a relationship between the knot positions for a stationary approximation to $f(x)$ using the H^1 semi-norm, in which we are interested, and the knot positions for a stationary approximation to $\mathrm{d}f/\mathrm{d}x$ in the L^2 norm using piecewise constant functions. Not surprisingly these turn out to satisfy the same equations. This is a useful result since it allows us to work in terms of approximation in the L^2 norm rather than the H^1 semi-norm.

The second lemma of this section is also an interesting result since it shows that provided the source term $r(x)$ is everywhere non-zero, then any steady MFE solution which has distinct knots cannot be degenerate. This is important because the local best approximation result of Theorem 2.1 does not guarantee this lack of degeneracy (due to parallelism).

We start then by showing the relation between our approximation to f and an L^2 approximation to $\mathrm{d}f/\mathrm{d}x$. In the proof of this lemma it is necessary to use the fact that the conventional Galerkin equations for this particular problem have a solution which is an interpolant of the solution of the PDE (see [25] for more details). This appears to be the main reason why it is difficult to extend the results of this section to a wider class of PDEs.

LEMMA 3.1 Let f be the unique steady solution of (3.1). The knot positions for a locally best first degree spline approximation to f in the H^1 semi-norm satisfy the same equations as the knot positions for a locally best zeroth degree spline approximation to $\mathrm{d}f/\mathrm{d}x$ in the L^2 norm.

Proof. For a stationary first degree spline approximation to f, equations (2.4) imply that

$$-\left(\frac{a_j - a_{j-1}}{s_j - s_{j-1}}\right)^2 + \left(\frac{a_{j+1} - a_j}{s_{j+1} - s_j}\right)^2 - 2\left\langle \frac{d^2 f}{dx^2}, \alpha_j \frac{\partial v}{\partial x}\right\rangle = 0,$$

i.e.

$$0 = -\left(\frac{a_j - a_{j-1}}{s_j - s_{j-1}}\right)^2 - 2\frac{(a_j - a_{j-1})}{(s_j - s_{j-1})^2}\left\{\left[(x - s_{j-1})\frac{df}{dx}\right]_{s_{j-1}}^{s_j} - \int_{s_{j-1}}^{s_j} \frac{df}{dx}\,dx\right\}$$
$$+\left(\frac{a_{j+1} - a_j}{s_{j+1} - s_j}\right)^2 - 2\frac{(a_{j+1} - a_j)}{(s_{j+1} - s_j)^2}\left\{\left[(s_{j+1} - x)\frac{df}{dx}\right]_{s_j}^{s_{j+1}} + \int_{s_j}^{s_{j+1}} \frac{df}{dx}\,dx\right\}, \tag{3.2}$$

for $j = 1, \dots, N$. Also, equations (2.2) in this case imply that, for $j = 1, \dots, N$, $a_j = f(s_j)$, and when this substitution is made in (3.2) we get

$$0 = \left(\frac{f(s_j) - f(s_{j-1})}{s_j - s_{j-1}}\right)^2 - 2\left(\frac{f(s_j) - f(s_{j-1})}{s_j - s_{j-1}}\right)\frac{df}{dx}(s_j)$$
$$-\left(\frac{f(s_{j+1}) - f(s_j)}{s_{j+1} - s_j}\right)^2 + 2\left(\frac{f(s_{j+1}) - f(s_j)}{s_{j+1} - s_j}\right)\frac{df}{dx}(s_j), \tag{3.3}$$

for $j = 1, \dots, N$.

For a best zeroth degree spline approximation to df/dx in the L^2 norm we must solve

$$0 = \frac{\partial}{\partial m_i}\left\{\sum_{j=1}^{N+1} \int_{s_{j-1}}^{s_j} \left(\frac{df}{dx}(x) - m_j\right)^2 dx\right\} \tag{3.4}$$

for $i = 1, \dots, N+1$, and

$$0 = \frac{\partial}{\partial s_i}\left\{\sum_{j=1}^{N+1} \int_{s_{j-1}}^{s_j} \left(\frac{df}{dx}(x) - m_j\right)^2 dx\right\}$$

for $i = 1, \dots, N$. The first of these sets of equations implies that, for $i = 1, \dots, N+1$, we have

$$m_i = \frac{f(s_i) - f(s_{i-1})}{s_i - s_{i-1}}, \tag{3.5}$$

whereas the second set of equations gives

$$0 = \frac{\partial}{\partial s_i}\left\{\int_0^1 \left(\frac{df}{dx}(x)\right)^2 dx + \sum_{j=1}^{N+1} [-2m_j(f(s_j) - f(s_{j-1})) + m_j^2(s_j - s_{j-1})]\right\}$$

for $i = 1, \dots, N$. This implies that

$$0 = -2m_i \frac{df}{dx}(s_i) + m_i^2 + 2m_{i+1}\frac{df}{dx}(s_i) - m_{i+1}^2, \tag{3.6}$$

for $i = 1, \dots, N$, so combining these last equations with equations (3.5) gives equations (3.3) as required. □

LEMMA 3.2 Suppose that, in (3.1), $r(x) \in C[0, 1] \neq 0$ $\forall x \in (0, 1)$. Let f be the unique steady solution of this equation. Then any stationary L^2 approximation to df/dx from the manifold of variable knot piecewise constant splines, that has distinct knots, has a finite jump discontinuity at each knot. That is, all of the knots are active.

Proof. Suppose that we have a local best approximation to df/dx which has distinct knots but for which knot k is inactive. Then, in the notation of the proof of Lemma 3.1, $m_k = m_{k+1}$, and from equation (3.5) we have

$$m_k = \frac{f(s_k) - f(s_{k-1})}{s_k - s_{k-1}} = \frac{f(s_{k+1}) - f(s_k)}{s_{k+1} - s_k} = m_{k+1}.$$

Hence, by the Mean Value Theorem, there exists $\sigma_k \in (s_{k-1}, s_k)$ and $\sigma_{k+1} \in (s_k, s_{k+1})$, such that

$$\frac{df}{dx}(\sigma_k) = \frac{df}{dx}(\sigma_{k+1}).$$

But this contradicts the assumption that df/dx is strictly monotonic on the open interval (which is equivalent to having $r(x) \neq 0$), and so such a local best approximation cannot exist. □

Lemma 3.2 tells us that any stationary approximation to df/dx with distinct knots must have all of the knots active, provided $r(x) \neq 0$ in $(0, 1)$. By Lemma 3.1 and equation (3.5) this means that any stationary linear spline approximation to f, in the H^1 semi-norm, which has distinct knots, must have a change in its gradient at each knot. Hence Theorem 2.1 implies that the corresponding steady MFE solution is not degenerate. Since there must be at least one stationary approximation to df/dx which has distinct knots (any globally best approximation for example) there must therefore exist at least one steady solution of the MFE equations which is not degenerate.

The requirement that $r(x) \neq 0$ in Lemma 3.2 is a necessary condition to ensure that all solutions of the stationary equations with distinct knots have active knots. It is not difficult to find functions $r(x)$, non-zero everywhere in $(0, 1)$ except at a single point, such that there exists a stationary approximation to df/dx from a manifold of free-knot constant splines which does not have fully active knots. For example, if $r(x) = 2x - 1$ and $N = 1$ then $m_1 = m_2 = 0$ and $s_1 = \frac{1}{2}$ gives such an approximation—which is a saddle point in (m_1, m_2, s_1)-space. The worst possible case occurs if $r(x) \equiv 0$ since the solution of the PDE (3.1) then tends to zero as $t \to \infty$. The corresponding MFE solutions therefore tend to degeneracy as $t \to \infty$. This case is investigated numerically in the next section.

It should be observed that by using Theorem 2.1 and Lemma 3.1 it is possible to prove results about the steady MFE solutions of equation (3.1) by considering a best approximation problem in the L^2 norm. This is the motivation behind Lemma 3.2 and is also the motivation for considering the following two theorems due to Chow [5], which are generalizations of earlier results in Barrow *et al.* [1].

THEOREM 3.3 Let $g \in C^k[0, 1]$ with $(d^k g/dx^k) > 0$ on $[0, 1]$, and suppose that $\log |(d^k g/dx^k)|$ is concave in $(0, 1)$. Then, for each positive integer N, g has a unique best $L^2[0, 1]$ approximation from P_N^k, the manifold of piecewise polynomials of order k (degree $k - 1$) with N free knots.

THEOREM 3.4 Let $g \in C^{k+3}[0, 1]$ with $(d^k g/dx^k) > 0$ on $[0, 1]$. Then there exists a positive integer N_0 such that, for each integer $N > N_0$, g has a unique best $L^2[0, 1]$ approximation from P_N^k.

In the case $k = 1$ these results refer to the uniqueness and eventual uniqueness of a best piecewise constant spline approximation to g. At first this may seem to be of little use to us since Theorem 2.1 and Lemma 3.1 only relate MFE solutions with solutions of the *stationary* equations for this approximation problem. However, Loach [17] suggests that the proofs of these theorems may still go through for any stationary solutions which have a strictly increasing sequence of knot positions. This is certainly the case when $k = 1$, and is illustrated for the first of the theorems in the appendix to this paper, where it has only been necessary to assume that $(dg/dx) > 0$ in the *open* interval $(0, 1)$.

Having made this observation, and using the further fact that the sign of g could be changed in the above theorems without affecting the results, we are now able to deduce the following two related theorems. In order to obtain them we have set $k = 1$ and $g = (df/dx)$. We then use the results of Lemma 3.1 and Theorem 2.1 to relate the positions of the node points in the Moving Finite Element method with the stationary knot points in the first order spline approximation to g. The final step is then to note that $(d^2f/dx^2) = -r(x)$.

THEOREM 3.5 Let $r \in C[0, 1]$ be non-zero in $(0, 1)$. Suppose that $\log|r|$ is concave in $(0, 1)$. Then for each positive integer, N, the MFE equations (with N free nodes) for problem (3.1), with Dirichlet boundary conditions, have a unique steady solution.

THEOREM 3.6 Let $r \in C^3[0, 1]$ be non-zero in $(0, 1)$. Then there exists a positive integer N_0 such that, for each integer $N > N_0$, the MFE equations (with N free nodes) for problem (3.1), with Dirichlet boundary conditions, have a unique steady solution.

An immediate corollary to these results comes from the fact that the unique best approximation to df/dx from P_N^1 is itself a stationary solution with a strictly increasing sequence of knot points. Hence, again using equations (3.5), the unique steady solutions of the above theorems will also be the unique best linear spline approximations to f in the H^1 semi-norm, satisfying the Dirichlet boundary conditions.

Note that Theorems 3.5 and 3.6 are given for equation (3.1) with arbitrary fixed Dirichlet boundary conditions rather than homogeneous conditions. This extension is perfectly straightforward and can in fact be generalized to a yet wider class of boundary conditions, as outlined in the previous section.

4. Numerical investigations

This section is divided into two parts. In the first subsection we consider the question of whether the steady MFE solutions deduced in the previous sections are analytically stable solutions of the semi-discrete Moving Finite Element equations. In the second subsection we investigate the behaviour of the MFE solutions to the linear heat equation in the absence of any source term, $r(x)$. In this case the steady solution of the PDE is always linear and so there is extreme non-uniqueness in the solution of the best approximation problem.

4.1 *Stability of steady solutions of the MFE equations*

So far in this paper we have deduced that for certain linear PDEs which possess unique, stable, steady solutions, the Moving Finite Element method has corresponding steady solutions. Moreover, these MFE solutions satisfy a local best approximation property and, in the particular case of the inhomogeneous heat equation, there is a unique steady MFE solution which is the globally best approximation to the steady solution of the PDE. However we have not considered the question of whether the MFE solutions that we have derived are themselves analytically stable and if so, what is their domain of attraction? Without this information the previous results are only of limited value because we do not know if the Moving Finite Element method in practice ever achieves any of its steady solutions.

In this subsection we make a numerical investigation of this problem. It is difficult to obtain any analytic results due to the fact that we cannot explicitly calculate any of the steady MFE solutions in the general case and because the manifold on which these solutions lie is not linear.

We start by considering an example of an equation which satisfies the hypotheses of Theorem 3.5:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\pi^2}{5}\sin(\pi x), \quad 0\leqslant x\leqslant 1, \quad 0\leqslant t, \tag{4.1}$$

so, in this case, $r(x)=(\pi^2/5)\sin(\pi x)$. The solution of this problem, with homogeneous Dirichlet boundary conditions, tends to $\frac{1}{5}\sin(\pi x)$ as $t\to\infty$. When the Moving Finite Element method is used for various choices of initial data the discrete solution in each case tends to the same function whenever the same number of knot points are used. In the case of 6 free knots for example, the final solution has

$$\begin{array}{llllll} s_1=0\cdot2118 & s_2=0\cdot3380 & s_3=0\cdot4474 & s_4=0\cdot5526 & s_5=0\cdot6620 & s_6=0\cdot7882 \\ a_1=0\cdot1235 & a_2=0\cdot1747 & a_3=0\cdot1973 & a_4=0\cdot1973 & a_5=0\cdot1747 & a_6=0\cdot1235. \end{array}$$

A straightforward computation, using a NAG library optimization routine, verifies that this is indeed a best approximation to the steady solution, $\frac{1}{5}\sin(\pi x)$, in the H^1 semi-norm. Figures 1 and 2 show how the knot points evolve to their final positions for two different choices of initial data: $\frac{1}{2}\sin(\pi x)$ and $x(x-1)(x-2)$ respectively.

In the case of this particular equation it is possible, with the aid of a symbolic computation package such as Mathematica [28], to prove the linear stability of the steady MFE solutions for small values of N. This confirms what we show here numerically, but does not give a general proof of stability for all N.

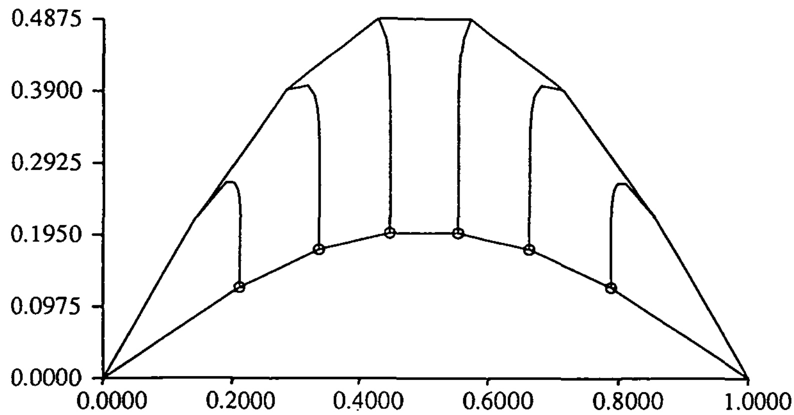

FIG. 1. Trajectories of the free knots in the MFE solution of equation (4.1) with initial data $\frac{1}{2}\sin(\pi x)$.

Similar results may be observed when other equations which satisfy the hypotheses of Theorem 3.5 are solved using the MFE method. Figures 3 and 4, for example, show the evolution of the knot positions for differing initial data in the case where

$$r(x)=\begin{cases}\dfrac{10x}{9} & 0\leqslant x\leqslant 0{\cdot}9\\ 10-10x & 0{\cdot}9\leqslant x\leqslant 1.\end{cases} \tag{4.2}$$

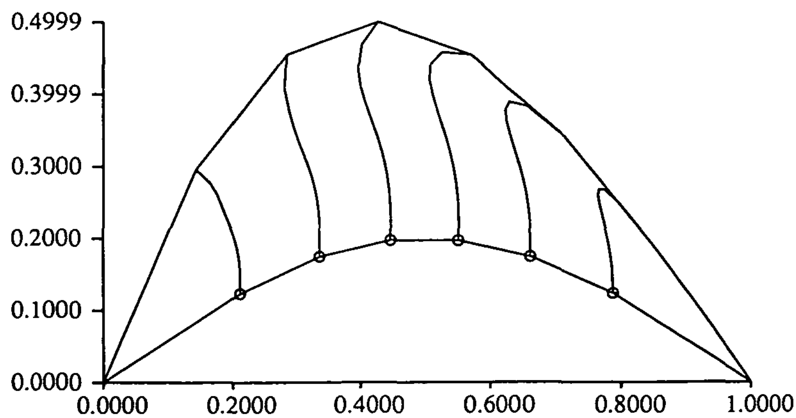

FIG. 2. Trajectories of the free knots in the MFE solution of equation (4.1) with initial data $x(x-1)(x-2)$.

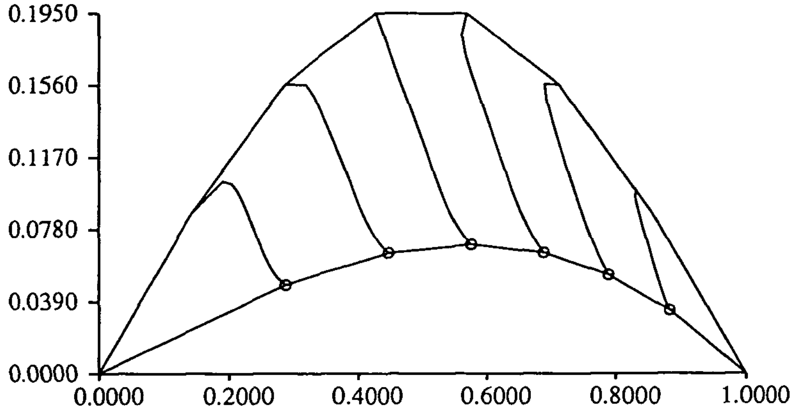

FIG. 3. Trajectories of the free knots in the MFE solution of equation (3.1) with $r(x)$ given by (4.2) and initial data $\frac{1}{2}\sin(\pi x)$.

Again the steady solution is stable and is the best approximation in the H^1 semi-norm.

For our final example in this subsection we consider the following equation:

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[x\frac{\partial u}{\partial x}\right]-\frac{16}{x}u, \quad 0\leqslant x\leqslant 1, \quad 0\leqslant t, \tag{4.3}$$

subject to the boundary conditions $u(0, t)=0$ and $u(1, t)=1$. Here $p(x)\equiv x$, $q(x)\equiv 16/x$ and the unique, stable, steady solution is $u(x, t)=x^4$. Strictly

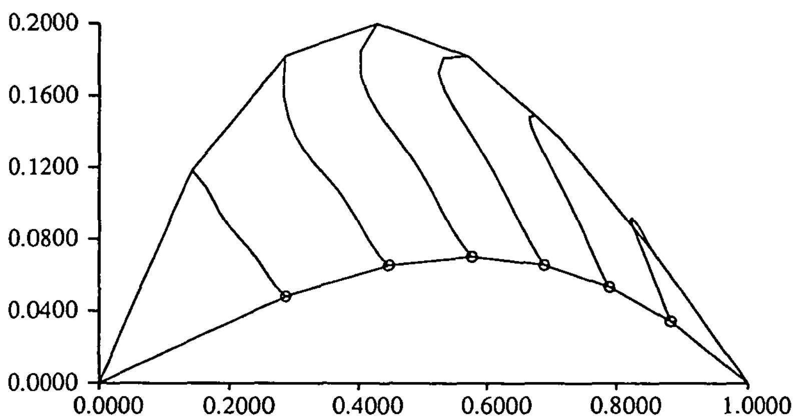

FIG. 4. Trajectories of the free knots in the MFE solution of equation (3.1) with $r(x)$ given by (4.2) and initial data $x(x-1)(x-2)$.

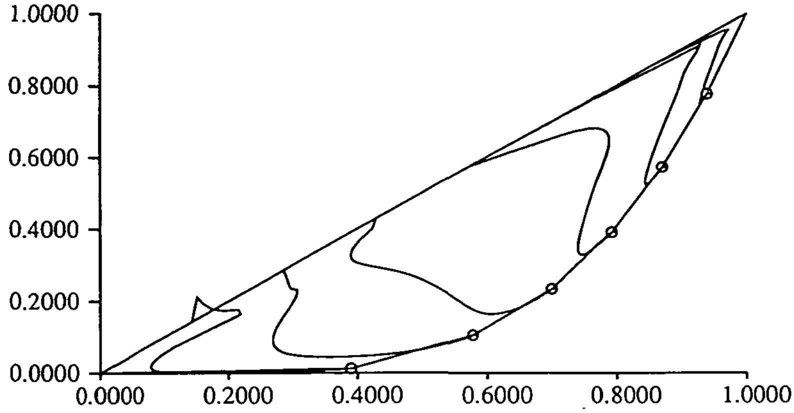

FIG. 5. Trajectories of the free knots in the MFE solution of equation (4.3) with linear initial data.

speaking this equation does not satisfy the hypotheses of Theorem 2.1 since $q(x)$ is not square integrable. However because of the homogeneous boundary condition at $x=0$ this does not actually cause any difficulties since the condition on $q(x)$ is only present to ensure existence and uniqueness of a steady solution to the PDE in the general case, it is not used explicitly in the proof of the theorem.

Figure 5 shows the Moving Finite Element solution of this problem subject to the initial condition $u(x, 0)=x$. Again it may be verified numerically that the steady state reached is indeed a local best approximation to the true steady solution, x^4, this time in the norm $\|\cdot\|$, given by

$$\|u\|=\left\{\int_0^1\left(x\left[\frac{\partial u}{\partial x}\right]^2+\frac{16}{x}u^2\right)\mathrm{d}x\right\}^{\frac{1}{2}}.$$

Unlike in the other two examples shown here however, it does seem possible to find choices of initial data for which the method does not converge to a steady solution. Instead the phenomenon of nodes merging together occurs, such as in the case where $u(x, 0)=\sin(\pi x/2)$. This suggests that the domain of attraction of the steady MFE solutions in this case is not as large. Nevertheless, in each example considered, a stable steady MFE solution does exist.

4.2 *Large time solutions of the homogeneous heat equation*

The main results of Section 3; Lemma 3.2 and Theorems 3.5 and 3.6, all require the heat equation, (3.1), to have a non-zero source term, $r(x)$. In this subsection we consider what happens in the special case where $r(x)\equiv 0$. In this instance the steady solution of the PDE is identically zero, which means that it has infinitely many piecewise linear approximants satisfying the stationary equations of Theorem 2.1. We will show that the positions which the knot points tend to as

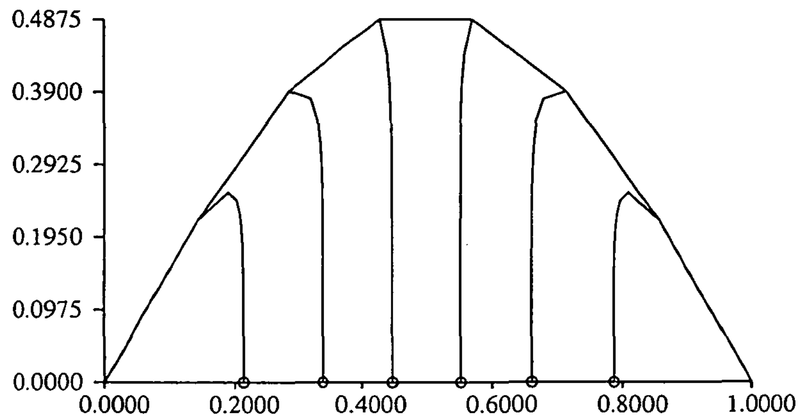

FIG. 6. Trajectories of the free knots in the MFE solution of equation (3.1) with $r(x) \equiv 0$ and initial data $\frac{1}{2}\sin(\pi x)$.

$t \to \infty$ for this equation seem to satisfy an alternative best approximation property, this time based on the temporal part of the solution.

When $r(x) \equiv 0$ the solution to equation (3.1) satisfies

$$u(x, t) \sim A_1 e^{-\pi^2 t} \sin(\pi x) \quad \text{as } t \to \infty, \tag{4.4}$$

provided $\int_0^1 u(x, 0) \sin(\pi x)\, dx \neq 0$. Due to the effects of rounding error it is to be expected that for all choices of initial data, $u(x, 0)$, (4.4) should be approximated by any acceptable numerical method. We will show that the Moving Finite Element method appears to choose its knot positions so as to give the best possible approximation to the temporal part of (4.4).

Figure 6 shows a typical solution to this problem, starting with initial data $u(x, 0) = \frac{1}{2}\sin(\pi x)$. As can be seen the knot positions tend to fixed values whilst the amplitudes diffuse to zero. In fact the rate of diffusion of these amplitudes also tends to a fixed value which depends only on the number of knots being used, N. This is illustrated in Table 1. It seems that this ultimate diffusion rate is tending to $-\pi^2$, monotonically from below, as $N \to \infty$. This in itself is a useful observation, although perhaps to be expected if the method is convergent. What is more interesting however is the connection between this ultimate diffusion rate, k say, and N.

Equations (1.5), with $p \equiv 1$, $q \equiv r \equiv 0$ and $\dot{\boldsymbol{s}} = \boldsymbol{0}$, give the standard Galerkin equations for this problem on a fixed spatial mesh, $\boldsymbol{s}$. We may look for solutions of these equations with the property

$$\dot{\boldsymbol{a}}(t) = k\boldsymbol{a}(t),$$

and so (1.5) may be expressed as

$$kM(\boldsymbol{s})\boldsymbol{a} = -K(\boldsymbol{s})\boldsymbol{a},$$

TABLE 1
Numerical calculations of the ultimate diffusion rate (k) *of the stable MFE solutions of equation* (3.1), *with homogeneous Dirichlet boundary conditions and* $r(x) \equiv 0$

Number of free knots (N)	Ultimate diffusion rate (k)	Difference between $-\pi^2$ and k
1	−12·000	2·130
2	−10·667	0·797
4	−10·135	0·265
8	−9·951	0·081
16	−9·895	0·025
32	−9·878	0·008
64	−9·872	0·002

where $M_{ij} = \langle \alpha_i, \alpha_j \rangle$ and $K_{ij} = \langle (\partial\alpha_i/\partial x), (\partial\alpha_j/\partial x) \rangle$ for $i, j = 1, \ldots, N$. Since both M and K are positive definite this is just a generalized eigenvalue problem whose solutions, k, are all negative. Hence we may define a continuous function $k(s)$ to be the smallest eigenvalue in modulus. It turns out that when this value is optimized over $s \in T$ for different values of N, it always comes out to be precisely the value given in Table 1, obtained by the MFE method. Moreover the corresponding maximizing knot positions are exactly those that the MFE solution tended to.

The implication of the above observations is that the MFE method for solving the homogeneous heat equation does tend to a solution with fixed knots and the position of these knots is such that the resulting solution diffuses at a rate which is as close as possible to the true rate of $-\pi^2$. Hence in the situation where the best spatial approximation problem is degenerate, the method appears to satisfy a best temporal criterion.

5. Conclusions

The Moving Finite Element method seems to be well suited to solving one-dimensional time-dependent partial differential equations which possess stable steady solutions, especially in the region of these solutions. For the class of linear problems considered here the MFE equations also possess steady solutions which appear to be stable and which satisfy the stationary equations for a best spatial approximation to the steady solution of the PDE. Since a globally best approximation to the steady solution of the PDE must itself satisfy the stationary equations this must also be one of the steady solutions of the MFE equations. In certain cases it may be demonstrated that the MFE method always gives the best possible spatial approximation to the steady solution and so, in this sense, it can be said that using the Moving Finite Element method is like using the standard Galerkin method but with the best possible choice of spatial mesh.

Initially there were thought to be two potential drawbacks with using the MFE algorithm without penalty functions, these being the problems of degeneracy due to parallelism (neighbouring elements having the same slope) and of nodes merging together. The first of these situations does not in fact seem to be a problem at all, since the algorithm appears to direct nodes along paths which lead to a best temporal approximation as $t \to \infty$. The second situation is more serious however, as the MFE solution in this case is clearly following a path towards a solution of the stationary equations which has coincident nodes. At the moment this causes the algorithm to break down, so it is necessary to obtain some way of either avoiding or overcoming this situation. The penalization technique of Miller ([20], [21] or [8]) is an avoidance strategy, however this necessarily has side effects on the approximation properties of the method. Perhaps a better approach would be somehow to overcome the difficulty by recognizing when it is going to occur and then realigning the nodes so as to allow them to follow trajectories which lead to a stationary approximation which has *distinct* knots.

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A. Appendix

In this appendix we show that Theorem 3.3 due to Chow [5] still holds, at least in the case $k = 1$, for any *local* best approximation to $g \in C^k[0, 1]$ from P_N^k. We will attempt to use a notation that is compatible with that of Chow since our proof closely follows Section 3 of his paper. Hence, let P^1 be the set of polynomials of order one (degree zero). Also let $\pi(g, [a, b])$ denote the unique solution of the stationary equation

$$\frac{\partial}{\partial m}\int_a^b (g(x) - m)^2\,\mathrm{d}x = 0.$$

So,

$$m = \frac{\displaystyle\int_a^b g(x)\,\mathrm{d}x}{b - a} = \pi(g, [a, b]).$$

Now consider any first order (zeroth degree) spline approximation to $g(x)$ in the L^2 norm which satisfies the corresponding stationary equations and whose knot positions are strictly increasing. Let $\boldsymbol{\sigma} \in \mathfrak{R}^N$ be the vector of these knot points. So $0 < \sigma_1 < \cdots < \sigma_N < 1$, and we will refer to $\boldsymbol{\sigma}$ as a stationary mesh.

LEMMA A.1 Suppose dg/dx is continuous and strictly positive on $(0, 1)$. Then $\pi(g, [\sigma_{i-1}, \sigma_i]) + \pi(g, [\sigma_i, \sigma_{i+1}] = 2g(\sigma_i)$ for $i = 1, \ldots, N$, where $\boldsymbol{\sigma}$ is a stationary mesh and, as usual, $\sigma_0 = 0$ and $\sigma_{N+1} = 1$.

Proof. We use the result (3.6), obtained in the proof of Lemma 3.1, where df/dx has been replaced by g:

$$-2m_i g(\sigma_i) + m_i^2 + 2m_{i+1} g(\sigma_i) - m_{i+1}^2 = 0.$$

From this we deduce that

$$2g(\sigma_i)(m_i - m_{i+1}) = (m_i + m_{i+1})(m_i - m_{i+1}).$$

By Lemma 3.2, again with df/dx replaced by g, we know that $m_i - m_{i+1} \neq 0$. Hence

$$2g(\sigma_i) = (m_i + m_{i+1})$$

for $i = 1, \ldots, N$. But from equation (3.4) we can easily show that $m_i = \pi(g, [\sigma_{i-1}, \sigma_i])$ and $m_{i+1} = \pi(g, [\sigma_i, \sigma_{i+1}])$, so the result is proved. □

Given g, we may now define $\boldsymbol{F} : T \to \Re^N$ such that $\boldsymbol{F}(\boldsymbol{s}, g) = (F_1(\boldsymbol{s}, g), \ldots, F_N(\boldsymbol{s}, g))$, by

$$F_i(\boldsymbol{s}, g) = 2g(s_i) - \pi(g, [s_{i-1}, s_i]) - \pi(g, [s_i, s_{i+1}])$$

for $i = 1, \ldots, N$. It may be observed that $\boldsymbol{F}$ has the following properties:

(a) $\boldsymbol{F}$ is continuous on the closure of T and differentiable in T.
(b) $\boldsymbol{F}$ vanishes at any stationary grid $\boldsymbol{\sigma}$.
(c) $\boldsymbol{F}$ has at least one zero in T (since there is at least one optimum grid and such a grid must be stationary).

For the following two results the proofs are almost identical to those given in Chow [5], the main differences being that we are only interested in polynomials of order one and grids in the interior of T. Also whenever Chow refers to an optimum grid we are of course only considering a stationary grid.

LEMMA A.2 Let $g \in C^1[0, 1]$ with $dg/dx > 0$ in $(0, 1)$. Suppose that $\log(dg/dx)$ is concave in $(0, 1)$. Then $\det(D\boldsymbol{F}) > 0$ if $\boldsymbol{F}(\boldsymbol{s}, g) = \boldsymbol{0}$.

THEOREM A.3 Let $g \in C^1[0, 1]$ with $dg/dx > 0$ in $(0, 1)$. Suppose that $\log(dg/dx)$ is concave in $(0, 1)$. Then for each positive integer N, g has a unique stationary $L^2[0, 1]$ approximation from the manifold of first order (piecewise constant) splines with distinct free knots.

The following corollary follows from Theorem 3.3 and the fact that a best approximation is also a stationary approximation.

COROLLARY A.4 The unique stationary $L^2[0, 1]$ approximation to g proved by the previous theorem is also the best approximation to g from the given manifold.