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# Adjoint Error Estimation for EHL-like Models

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SUMMARY

The calculation of friction when solving elastohydrodynamic lubrication (EHL) problems is of considerable practical engineering importance. Adjoint techniques allow the error in this integral quantity to be estimated and controlled as part of an adaptive solution strategy. This paper considers two simplified EHL models and demonstrates the successful application of the adjoint approach to error estimation of friction-like quantities for this challenging class of problem. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: Adjoint error estimation, elastohydrodynamic lubrication, friction

## 1. INTRODUCTION

Elastohydrodynamic lubrication (EHL) occurs when a lubricant flows between non-conformal machine components under extremely heavy loads. The pressures generated are sufficient to alter the rheology of the lubricant to that of a glass or plastic, and in such conditions the machine components deform elastically. The computation of this highly nonlinear free boundary problem may require a large number of points on uniform meshes. For example, [1] shows how $100,000$ mesh points may be necessary in one dimension to resolve fully all the features of interest. Use of non-uniform meshes for EHL solutions has been investigated previously, e.g. [1, 2] among others, but they have not commonly been applied in practice.

An important industrial requirement of EHL calculations is to be able to predict accurately the friction generated through a contact, for a given lubricant and a given set of operating conditions. The friction is an integral quantity that depends on the pressure derivative, the film thickness and the viscosity of the oil. Furthermore, it is desirable that the solution be calculated as efficiently as possible subject to the constraint that the resulting friction is estimated sufficiently accurately. This provides the motivation for the application of adjoint error estimation techniques, combined with local mesh refinement, explored in this paper.

The mathematical model describing a one-dimensional steady state line contact is given in [3]. This consists of a second order equation for the pressure ($p$) and two further equations

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involving the film thickness ($h$), the applied load ($w$), and the unknown cavitation point $X_c$. In dimensional form the equations are:

$$\frac{\partial}{\partial x}\left(\frac{\rho h^3}{\eta}\frac{\partial p}{\partial x}\right) - 6u_s\frac{\partial(\rho h)}{\partial x} = 0, \tag{1}$$

$$h(x) = h_0 + \frac{x^2}{2R} - \frac{4}{\pi E'}\int_{-\infty}^{X_c} \ln\left|\frac{x-x'}{x_0}\right| p(x')\,\mathrm{d}x', \qquad \int_{-\infty}^{X_c} p(x)\,\mathrm{d}x = w, \tag{2}$$

where $u_s$, $R$, $E'$ and $x_0$ are given constants, $p$, $h_0$ and $X_c$ are to be determined, and the viscosity ($\eta$) and density ($\rho$) vary nonlinearly with $p$.

In this paper we consider two simplified versions of the system (1)-(2), which are motivated and described in the following section. Section 3 then briefly outlines the key ideas behind the adjoint-based error estimate presented here, whilst Section 4 presents some illustrative results when the adjoint procedure is applied to the two model problems and considers how it can be used to drive adaptivity. Finally, Section 5 briefly discusses the success of this approach and its possible future extensions.

## 2. SIMPLIFIED PROBLEM

For the purposes of understanding and controlling the discretisation error for the full EHL problem, we consider two simplified line contact models, one of which is linear and the other nonlinear. In both cases, we consider an isoviscous, incompressible lubricant, where the surface geometry does not depend explicitly on the pressure.

The simplest model that we consider here involves solving a two point boundary value problem, given by

$$\frac{d}{dX}\left(H^3\frac{dP}{dX}\right) - \lambda\frac{dH}{dX} = 0, \tag{3}$$

$$H(X) = H_0 + \frac{X^2}{2}, \tag{4}$$

with boundary conditions $P(X_{-\infty}) = P(X_c) = 0$. The position of the right hand boundary, $X_c$, is fixed at a pre-determined value, as is $H_0$. The other boundary position, $X_{-\infty}$, is chosen to be sufficiently far to the left such that the friction is insensitive to changes in its value.

This linear problem may be modified so as to be closer to (1)-(2), by allowing $X_c$ to be an unknown free boundary and by treating $H_0$ as a further unknown, to be determined by constraining the pressures to balance an applied load. This yields the following additional equation and boundary condition:

$$\int_{X_{-\infty}}^{X_c} P(X)\,\mathrm{d}X = L, \qquad P'(X_c) = 0, \tag{5}$$

where $X_{-\infty}$ is defined to be equal to $X_c$ minus a given, constant, domain size.

A finite difference discretisation of the type typically employed in EHL calculations, [4], has been used in this work. This consists of a second order scheme based upon a central difference approximation to $\frac{d}{dX}(H^3\frac{dP}{dX})$ in equation (3) given above.

## 3. ADJOINT THEORY

The adjoint approach used here is based upon the ideas presented in [5] and [6], although there are a number of computational details that are problem dependent. Following the approach of [5], the subscript $H$ denotes quantities computed on an existing "coarse" mesh. Similarly, the subscript $h$ denotes quantities on a uniform refinement of this mesh, and $I_h^H$ denotes cubic spline interpolation of the coarse mesh values onto thsi "fine" mesh.

Given a computed coarse mesh solution, $u_H$, and a corresponding fine mesh functional, $F_h(I_h^H u_H)$, an estimate of the error in this functional may be obtained by predicting what its value would be were the solution, $u_h$, to be calculated on the fine mesh: $F_h(u_h)$. Whilst this does not directly give an estimate of the total error, by refining the mesh until the difference between consecutive meshes is sufficiently small, the error can be controlled. This is achieved through solving an extra system of equations on the coarse mesh:

$$\left[\frac{\partial R_H}{\partial u_H}\right]^T \Psi_H = \left(\frac{\partial F_H}{\partial u_H}\right)^T, \tag{6}$$

where $R$ is the system of finite difference residual equations, $u$ is the set of dependent variables and $F$ is the functional of interest. The solution $\Psi_H$ is the adjoint vector. This gives the sensitivity of the functional to the residuals which, along with the solution $u_H$, may be interpolated onto the fine mesh to calculate a correction as shown:

$$\tilde{F}(u_H) = F_h(I_h^H u_H) - (I_h^H \Psi_H)^T R_h(I_h^H u_H). \tag{7}$$

We refer to $(I_h^H \Psi_H)^T R_h(I_h^H u_H)$, in equation (7) as the correction, since this is the quantity which is used to "correct" the friction calculated from the coarse mesh solution. This correction provides an estimate of $F_h(I_h^H u_H) - F_h(u_h)$: hence $\tilde{F}$ represents an estimate of $F_h(u_h)$. The correction will also be used as the basis for adaptive refinement by refining the mesh in the areas where the contribution to this correction is largest, i.e. where the residuals weighted by scaling factors from the adjoint solution are large.

As indicated in the introduction, the functional that is of interest in this work is the friction which, for our model problems, is given by

$$F(P) = \int_{X_{-\infty}}^{X_c} -\frac{\partial P}{\partial X}\frac{H}{2} + \frac{\overline{\eta}}{H}\, \mathrm{d}X. \tag{8}$$

The first term in the integrand specifies the rolling contribution to the friction, whilst the second term represents the sliding component within the contact. The $\frac{\partial P}{\partial X}$ term is discretised using a central difference scheme, and the integral is approximated numerically using the trapezoidal rule. The derivative of this discrete version of (8) forms the right-hand side of (6).

It should also be noted that the presence of $X_c$ and $H_0$ in the friction calculation, (8), must be accounted for in the adjoint formulation of the nonlinear model problem for which they are both unknowns. Hence, in the notation of (6), $R_H$ consists of residuals for the finite difference approximation of $P$ at each interior mesh point plus two residuals obtained from discrete forms of the equations in (5), shown here as

$$R_{H_0} = L - \int_{X_{-\infty}}^{X_c} P(X)\, \mathrm{d}X, \qquad\qquad R_{X_c} = -P'(X_c).$$

Table I. Error estimates for uniform meshes and the ratio to the actual error ($\overline{\eta} = 0.1$)

| No. mesh points | Interpolated friction | Calculated correction | Corrected friction | True friction | Measured error | Effect. index |
|---|---|---|---|---|---|---|
| 65 | 4.5181307 | 0.1493361 | 4.3687946 | 4.3318657 | 0.1862650 | 1.2472 |
| 129 | 4.3387398 | -0.0047813 | 4.3435212 | 4.3496297 | -0.0108899 | 2.2775 |
| 257 | 4.3504149 | -0.0047481 | 4.3551631 | 4.3554793 | -0.0050643 | 1.0665 |
| 513 | 4.3556633 | -0.0014459 | 4.3571091 | 4.3571718 | -0.0015085 | 1.0433 |
| 1025 | 4.3572185 | -0.0003975 | 4.3576160 | 4.3576227 | -0.0004042 | 1.0169 |
| 2049 | 4.3576345 | -0.0001030 | 4.3577374 | 4.3577381 | -0.0001036 | 1.0065 |
| 4097 | 4.3577411 | -0.0000261 | 4.3577672 | 4.3577672 | -0.0000262 | 1.0028 |
| 8193 | 4.3577680 | -0.0000066 | 4.3577745 | 4.3577745 | -0.0000066 | 1.0019 |

Similarly, $u_H$ consists of the unknown pressure at each of the mesh points as well as $X_c$ and $H_0$. In this case, the Jacobian matrix has a tridiagonal structure plus two non-zero rows (corresponding to ($\frac{\partial R_{H_0}}{\partial P_i}$, $\frac{\partial R_{X_c}}{\partial P_i}$) and two further non-zero columns corresponding to ($\frac{\partial R_{p_i}}{\partial H_0}$, $\frac{\partial R_{p_i}}{\partial X_c}$) ). The Jacobian matrix thus has an arrowhead structure. Having obtained the solution to this full adjoint approximation, the calculation of the estimated friction, $F_h(I_h^H)$, is performed using equation (7).

## 4. RESULTS

In this section we present selected results for the adjoint error correction applied to both the nonlinear and linear problems respectively.

The first column of Table I indicates the number of mesh points used in the solution of the coarse mesh problem. Column 2 indicates the friction as calculated using the interpolant of this solution on the uniform refinement of this mesh. Column 3 shows the estimated correction to this friction value (computed according to (7)), and column 4 shows the corrected value. Column 5 gives the friction value had it been calculated on the uniformly refined mesh, with column 6 giving the measured error between columns 5 and 2. The ratio of the correction to the measured error (the Effectivity index) is shown in column 7, which can be seen to be converging to unity with increasing mesh resolution. This shows how the linear approximation becomes more accurate as the non-linear contributions decrease with increased mesh refinement.

Having established the accuracy of the error estimate on uniform meshes, Table II shows similar results for the same nonlinear problem when solved using non-uniform meshes. The starting mesh is refined by one level in the right half of the mesh, and a further level in the final quarter of the mesh. Again, we evaluate the error estimate against the actual error as measured between each non-uniform mesh and one which is a uniform refinement of it. The quality of the adjoint estimates on these irregular spaced meshes is shown to be just as good as those obtained on uniform grids. This is clearly of great importance if this technique is to be used to control local mesh refinement in a reliable manner.

Finally in this section we consider a simple adaptive mesh refinement strategy for the linear

Table II. Error estimates for non-uniform meshes and the ratio to the actual error ($\overline{\eta} = 0.1$)

| No. mesh points | Interpolated friction | Calculated correction | Corrected friction | True friction | Measured error | Effect. index |
|---|---|---|---|---|---|---|
| 33 | 4.5187870 | 0.1490240 | 4.3697630 | 4.3327976 | 0.1859894 | 1.2480 |
| 65 | 4.3389067 | -0.0048500 | 4.3437567 | 4.3498617 | -0.0109550 | 2.2587 |
| 129 | 4.3504571 | -0.0047639 | 4.3552210 | 4.3555374 | -0.0050803 | 1.0664 |
| 257 | 4.3556739 | -0.0014496 | 4.3571235 | 4.3571864 | -0.0015124 | 1.0433 |
| 513 | 4.3572212 | -0.0003984 | 4.3576196 | 4.3576264 | -0.0004052 | 1.0169 |
| 1025 | 4.3576351 | -0.0001032 | 4.3577384 | 4.3577390 | -0.0001039 | 1.0066 |
| 2049 | 4.3577412 | -0.0000262 | 4.3577674 | 4.3577675 | -0.0000262 | 1.0028 |
| 4097 | 4.3577680 | -0.0000066 | 4.3577746 | 4.3577746 | -0.0000066 | 1.0019 |

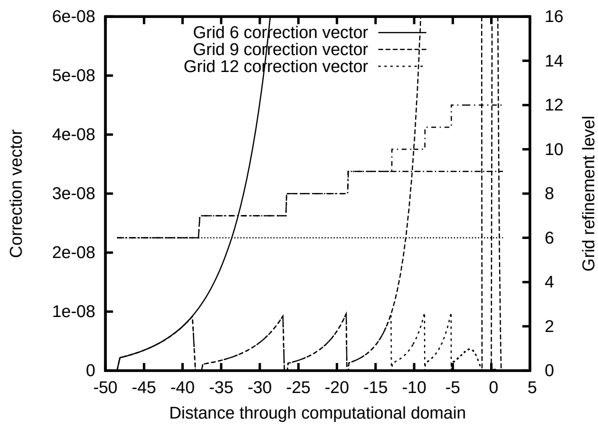


Figure 1. Correction vector with different levels of mesh refinement, right hand axis

model problem. Local mesh bisection is carried out where the *components* of the correction term defined by (7) exceed a prescribed tolerance. The correction is calculated using the coarse mesh solutions of $P$ and $\Psi$ interpolated onto the fine mesh. Figure 1 shows the refined areas for a sequence of meshes automatically adapted based upon a tolerance of $10^{-8}$. As the mesh is refined, the components of the correction term in the refined area are reduced. This continues until the components of the correction for the resulting adaptive mesh are roughly equal (and always less than the tolerance) throughout the domain. Figure 2 shows how an increasingly strict tolerance for the correction term results in an increase in the accuracy of the friction calculation, but with fewer mesh points.

## 5. CONCLUSIONS AND FUTURE WORK

In this work we have considered the extension of adjoint-based error estimation techniques to model problems representing some of the key elements of EHL cases. The friction through

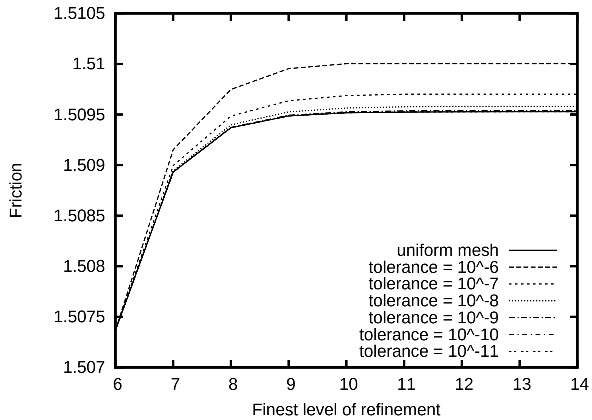


Figure 2. Convergence of friction with maximum allowed refinement level for different tolerances

the contact has been used for the functional since this is an industrially relevant quantity of interest. The free boundary in these cases has also been included in the adjoint system. The results have shown that this method provides excellent predictions for inter-grid friction error using both uniform and non-uniform meshes. Moreover, we have shown how the correction term can be used to highlight the areas of the domain in which the solution contributes the most to the inter-grid friction error, and can therefore be used as a basis for local refinement.

Although these early results are promising it is also clear that a more sophisticated refinement procedure, such as those presented in [5] or [7], may pay dividends. In the former, the idea of a "duality gap" is introduced which indicates the areas with the greatest nonlinear influence between the grids. If this can be reduced, the nonlinear contribution to the error can also be reduced, which would clearly be of use in this work. In continuing work the extension of this approach to the full steady state EHL-line case is being undertaken.

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