

# Adaptive finite element solutions of time-dependent partial differential equations using moving mesh algorithms

Mike Baines

Dept of Mathematics, University of Reading, UK

Matthew Hubbard, Peter K. Jimack

School of Computing, University of Leeds, UK

*presented at the Workshop on Modelling and Simulation in Chemical Engineering, Coimbra, Portugal, June 30th- July 4th 2003*

## Abstract

A Lagrangian moving finite element algorithm is presented whose mesh velocity is determined by the invariance of the local "mass". The method is applied to second and fourth order nonlinear diffusion equations with moving boundaries in one and two dimensions.

## 1 Introduction

We consider adaptive finite element solutions of second order and fourth order nonlinear diffusion equations with moving boundaries using a Lagrangian moving finite element method. The method is prompted by recent interest in geometric integration and scale invariance (for references see [6]) which has rekindled interest in the use of adaptive moving meshes in the solution of these equations, suggesting new numerical approaches. The invariance properties combine independent and dependent variables, suggesting that these variables should be treated similarly in numerical work. A natural consequence is to use moving adaptive meshes.

Scale invariance implies a local relationship between the variables which can be used to drive mesh movement. The mechanism is similar to the use of monitor functions to control the movement of the mesh, as in the MMPDE (Moving Mesh Partial Differential Equations) method [9]. It is also related to the Geometric Conservation Law [11] and its associated invariance property

[7]. The local relationship induces a mesh movement which retains the scale invariance properties of the original PDE.

The method is a generalisation of the finite volume approach used in [3, 4] in one dimension. It uses a weighted form of the invariance equation on a patch of elements, as in [1], resulting in a Lagrangian moving finite element method. In order to obtain uniqueness in higher dimensions we exploit the idea of a mesh velocity potential, as proposed in [7]. The link between the method and classical fluid dynamics is discussed in [2].

We describe the Lagrangian moving finite element method and its role in free boundary problems requiring adaptivity. The method is tested against the radial self-similar solution of the two-dimensional Porous Medium Equation (PME) with a free boundary. We also consider applications to problems governed by a fourth order nonlinear diffusion equation with a moving boundary.

We begin by setting up a moving framework for the theory.

## 2 Fixed and Moving Frames

Consider a scalar PDE in the general form

$$\frac{\partial u}{\partial t} = Lu \tag{1}$$

where  $u = u(\mathbf{x}, t)$  in a fixed frame of reference with coordinate  $\mathbf{x}$  and  $L$  is a multidimensional operator involving space derivatives only.

Instead of working in the fixed frame we take a Lagrangian viewpoint. Define an invertible mapping between fixed labelling coordinates  $\mathbf{a}$  at time  $\tau$  and moving coordinates  $\mathbf{x}$  at time  $t$ , of the form

$$\mathbf{x} = \hat{\mathbf{x}}(\mathbf{a}, \tau), \quad t = \tau$$

so that

$$u(\mathbf{x}, t) = u(\hat{\mathbf{x}}(\mathbf{a}, \tau), \tau) = \hat{u}(\mathbf{a}, \tau)$$

say, where  $\hat{u}$ ,  $\hat{\mathbf{x}}$  are Eulerian coordinates.

By the chain rule,

$$\frac{\partial \hat{u}}{\partial \tau} = \nabla u \cdot \frac{\partial \hat{\mathbf{x}}}{\partial \tau} + \frac{\partial u}{\partial t} \frac{dt}{d\tau}$$

where  $\frac{\partial}{\partial t}$  means differentiation with respect to time  $t$  with  $\mathbf{x}$  frozen, so that  $\frac{\partial u}{\partial t}$  is given by the PDE (1). Hence, writing

$$\dot{u} = \frac{\partial \hat{u}}{\partial \tau}, \quad \dot{\mathbf{x}} = \frac{\partial \hat{\mathbf{x}}}{\partial \tau},$$

we obtain the form of the differential equation in the moving frame,

$$\dot{u} - \nabla u \cdot \dot{\mathbf{x}} = Lu. \quad (2)$$

Clearly, a second equation is required to determine the two unknown Eulerian velocities  $\dot{u}$  and  $\dot{\mathbf{x}}$ .

An integral form similar to (2) may be obtained using Leibnitz' rule on a moving test volume  $\Omega(t)$  in the form

$$\frac{d}{dt} \int_{\Omega(t)} u d\Omega = \frac{\partial}{\partial t} \int_{\Omega(t)} u d\Omega + \oint_{\partial\Omega(t)} u \dot{\mathbf{x}} \cdot d\mathbf{\Gamma} = \int_{\Omega(t)} \left( \frac{\partial u}{\partial t} + \nabla \cdot (u \dot{\mathbf{x}}) \right) d\Omega \quad (3)$$

giving the integral form in the moving frame (cf. (2)),

$$\frac{d}{dt} \int_{\Omega(t)} u d\Omega - \int_{\Omega(t)} \nabla \cdot (u \dot{\mathbf{x}}) d\Omega = \int_{\Omega(t)} Lu d\Omega \quad (4)$$

where we have made use of the integral form of the PDE (1) in the fixed frame.

For the finite element method we need weak forms. Given a set of suitable test functions  $w_i$  moving with the velocity field  $\dot{\mathbf{x}}$  a generalisation of the Leibnitz rule (3) gives

$$\begin{aligned} \frac{d}{dt} \int_{\Omega(t)} w_i u d\Omega &= \frac{\partial}{\partial t} \int_{\Omega(t)} w_i u d\Omega + \oint_{\partial\Omega(t)} w_i u \dot{\mathbf{x}} \cdot d\mathbf{\Gamma} \\ &= \int_{\Omega(t)} \left( w_i \frac{\partial u}{\partial t} + u \frac{\partial w_i}{\partial t} + \nabla \cdot (w_i u \dot{\mathbf{x}}) \right) d\Omega \end{aligned}$$

Assuming that the test functions  $w_i$  are advected with velocity  $\dot{\mathbf{x}}$ , we have

$$\frac{\partial w_i}{\partial t} + \dot{\mathbf{x}} \cdot \nabla w_i = 0 \quad (5)$$

leading to the integral weak form in the moving frame,

$$\frac{d}{dt} \int_{\Omega(t)} w_i u d\Omega - \int_{\Omega(t)} w_i \nabla \cdot (u \dot{\mathbf{x}}) d\Omega = \int_{\Omega(t)} w_i Lu d\Omega \quad (6)$$

where we have made use of the weak form of the PDE (1) in the fixed frame.

### 3 A Distributed Conservation Principle

Assume that the problem and the boundary conditions are such that the total mass

$$\int_{\Omega(t)} u d\Omega$$

is conserved in the moving frame. Motivated by the scale invariance of this quantity, we assume that the velocity  $\dot{\mathbf{x}}$  of the moving frame is determined locally by the distributed conservation principle

$$\int_{\Omega(t)} w_i u d\Omega = c_i = \text{constant in time.} \quad (7)$$

Differentiating (7) with respect to time,

$$\frac{d}{dt} \int_{\Omega(t)} w_i u d\Omega = 0$$

giving from (6)

$$- \int_{\Omega(t)} w_i \nabla \cdot (u \dot{\mathbf{x}}) d\Omega = \int_{\Omega(t)} w_i L u d\Omega \quad (8)$$

or, after integration by parts,

$$- \oint_{\partial\Omega(t)} w_i u \dot{\mathbf{x}} \cdot d\Gamma + \int_{\Omega(t)} u \dot{\mathbf{x}} \cdot \nabla w_i d\Omega = \int_{\Omega(t)} w_i L u d\Omega. \quad (9)$$

Equation (8) is in effect an equation for the divergence of  $u \dot{\mathbf{x}}$ . It is insufficient by itself to determine  $\dot{\mathbf{x}}$  uniquely but if the vorticity  $\text{curl} \dot{\mathbf{x}}$  is specified (together with a suitable boundary condition) then, given  $u$ , equations (8) or (9) determine the velocity  $\dot{\mathbf{x}}$ .

For example, suppose that  $\text{curl} \dot{\mathbf{x}} = \text{curl} \mathbf{q}$  is specified. Then there exists a velocity potential  $\phi$  such that

$$\dot{\mathbf{x}} = \mathbf{q} + \nabla \phi \quad (10)$$

so that (8) can be written

$$- \int_{\Omega(t)} w_i \nabla \cdot (u \nabla \phi) d\Omega = \int_{\Omega(t)} w_i (L u + \nabla \cdot (u \mathbf{q})) d\Omega \quad (11)$$

and (9) becomes

$$\begin{aligned}
& - \oint_{\partial\Omega(t)} w_i u \nabla \phi \cdot d\mathbf{\Gamma} + \int_{\Omega(t)} u \nabla \phi \cdot \nabla w_i d\Omega \\
& = \int_{\Omega(t)} w_i L u d\Omega + \oint_{\partial\Omega(t)} w_i u \mathbf{q} \cdot d\mathbf{\Gamma} - \int_{\Omega(t)} u \mathbf{q} \cdot \nabla w_i d\Omega. \quad (12)
\end{aligned}$$

Equation (12) can be used to determine  $\phi$ , after which  $\dot{\mathbf{x}}$  is given by the weak form, from (10),

$$\int_{\Omega(t)} w_i \dot{\mathbf{x}} d\Omega = \int_{\Omega(t)} w_i \nabla \phi d\Omega + \int_{\Omega(t)} w_i \mathbf{q} d\Omega. \quad (13)$$

## 4 A Moving Finite Element Method

A Moving Finite Element method may be constructed using the weak forms (7), (12) and (13).

Linear elements are used for  $u$ ,  $\dot{\mathbf{x}}$ , and  $\phi$ , here denoted by upper case  $U$ ,  $\dot{\mathbf{X}}$ , and  $\Phi$ , on a (moving) triangulation of the region. Since  $\dot{\mathbf{X}}$  is piecewise linear and  $W_i$  is the advected form of  $\hat{W}_i$  (cf. (5)), the corresponding functions  $W_i$  are the usual linear basis functions on the moving mesh. The support of the integrals in the  $i$ 'th equation (12) is taken to be the patch of elements  $\Pi_i(\dot{\mathbf{X}})$  surrounding the node. The Dirichlet condition  $U = 0$  is not imposed strongly at the boundary in the solution of (7), but weakly in the first term of (12).

In effect we solve the ODE system

$$\frac{d}{dt} \vec{\mathbf{X}} = \vec{\mathbf{F}}(\vec{\mathbf{X}}) \quad (14)$$

using the following sequence to evaluate  $\vec{\mathbf{F}}(\vec{\mathbf{X}})$ :

- Given  $\vec{\mathbf{X}}$  recover  $U$  from (7)
- Given  $U$  calculate  $\vec{\Phi}$  from (12)
- Calculate  $\vec{\mathbf{F}}(\vec{\mathbf{X}})$  from (13)
- Return

The overall algorithm requires the solution of a single ODE system, where each evaluation of the right-hand side of (14) involves the solution of three (four in the case of a fourth order problem, as outlined below) symmetric linear algebraic systems.

## 4.1 A Second Order Problem

The Porous Medium Equation in a fixed coordinate system (PME)

$$\frac{\partial u}{\partial t} = \nabla \cdot (u^m \nabla u), \quad (15)$$

is a well-known model equation for gas flows in porous media, spreading liquids etc. It admits compact support solutions with a free boundary for which comparison results are known [6, 10]. In integral form (15) is

$$\frac{d}{dt} \int_{\Omega(t)} u d\Omega = \int_{\Omega(t)} \nabla \cdot (u^m \nabla u) d\Omega = \oint_{\partial\Omega(t)} (u^m \nabla u) \cdot d\mathbf{\Gamma} \quad (16)$$

so that if  $u^m \nabla u$  vanishes on the boundary the total mass is conserved, *i.e.*

$$\int_{\Omega(t)} u r^{d-1} dr = \text{constant in time.} \quad (17)$$

Note that for this particular problem (12), with  $\mathbf{q}$  set to zero, becomes

$$\begin{aligned} & - \oint_{\partial\Omega(t)} w_i u \nabla \phi \cdot d\mathbf{\Gamma} + \int_{\Omega(t)} u \nabla \phi \cdot \nabla w_i d\Omega \\ &= - \int_{\Omega(t)} u^m \nabla w_i \cdot \nabla u d\Omega, \end{aligned} \quad (18)$$

where it is again assumed that  $u^m \nabla u$  vanishes on the boundary.

The radially symmetric form of (15) (in  $d$  dimensions) is

$$\frac{\partial u}{\partial t} = \frac{1}{r^{d-1}} \frac{\partial}{\partial r} \left( r^{d-1} u^m \frac{\partial u}{\partial r} \right). \quad (19)$$

Equation (19) is invariant under the scalings

$$t \rightarrow \lambda t, \quad r \rightarrow \lambda^\beta r, \quad u \rightarrow \lambda^{(2\beta-1)/m} u. \quad (20)$$

If the boundary conditions are such that the total mass is invariant, then it follows that  $\beta = 1/(2 + md)$ . A source-type self-similar solution,

$$u_{SS} = \begin{cases} \left(\frac{t_0}{t}\right)^{d/(2+md)} \left(1 - \left(\frac{r^2}{K t^{2/(2+md)}}\right)\right)^{1/m} & r^2 \leq K t^{2/(2+md)} \\ 0 & r^2 > K t^{2/(2+md)} \end{cases} \quad (21)$$

may be deduced from (20) [6], where  $t_0, K$  are constants, which may be used to test numerical results. The function  $u_{SS}$  vanishes at the moving boundary and a typical profile is sketched in cross-section in Figure 1 for  $m > 1$ . For  $m = 1$  the slope at the boundary is finite while for  $m > 1$  it is infinite. Recall that the global mass is conserved.

For this equation there also exists the following comparison principle [10]: given three sets of initial conditions,

$$u_1(x, y, t_0) \leq u_2(x, y, t_0) \leq u_3(x, y, t_0) \quad \forall (x, y) \in \Omega, \quad (22)$$

then

$$u_1(x, y, t) \leq u_2(x, y, t) \leq u_3(x, y, t) \quad \forall (x, y) \in \Omega, t \geq t_0. \quad (23)$$

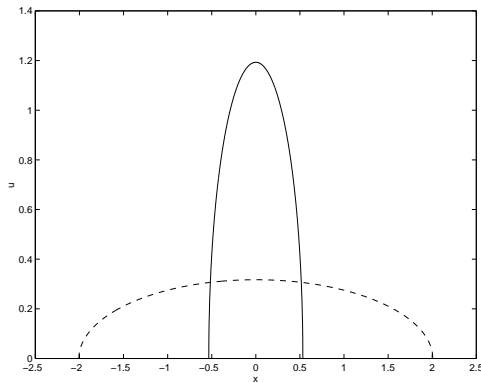


Figure 1: Typical behaviour of the self-similar solution of the PME.

## 4.2 A Fourth Order Problem

A corresponding fourth order equation (in the fixed coordinate system) is

$$\frac{\partial u}{\partial t} + \nabla \cdot (u^m \nabla (\nabla^2 u)) = 0 \quad (24)$$

which arises in the flow of surface-tension dominated thin liquid films ( $m = 3$ ) and the diffusion of dopant in semiconductors. In integral form it is

$$\frac{d}{dt} \int_{\Omega(t)} u d\Omega = - \int_{\Omega(t)} \nabla \cdot (u^m \nabla (\nabla^2 u)) d\Omega = - \oint_{\partial\Omega(t)} (u^m \nabla (\nabla^2 u)) \cdot d\Gamma = 0 \quad (25)$$

so that if  $u^m \nabla(\nabla^2 u)$  vanishes on the boundary the total mass is constant in time (cf. (17)). The equation (24) may be physically split up into the pair of second order equations

$$u_t + \nabla \cdot (u^m \nabla p) = 0, \quad p = \nabla^2 u \quad (26)$$

where  $p$  is a pressure. In this case, instead of (12), with  $\mathbf{q}$  set to zero, we now have

$$\begin{aligned} & - \oint_{\partial\Omega(t)} w_i u \nabla \phi \cdot d\mathbf{\Gamma} + \int_{\Omega(t)} u \nabla \phi \cdot \nabla w_i d\Omega \\ = & - \int_{\Omega(t)} u^m \nabla w_i \cdot \nabla \pi d\Omega, \end{aligned} \quad (27)$$

again assuming that  $u^m \nabla(\nabla^2 u)$  vanishes on the boundary. In (27)  $\pi$  is a weak approximation to the pressure given by

$$\int_{\Omega(t)} w_i \pi d\Omega = - \int_{\Omega(t)} \nabla w_i \cdot \nabla u d\Omega, \quad (28)$$

where an additional boundary condition, stating that the normal derivative of  $u$  is zero throughout  $\partial\Omega(t)$ , has been used.

The fourth order radially symmetric equation (24) in  $d$  dimensions (in split form) is

$$\frac{\partial u}{\partial t} + \frac{1}{r^{d-1}} \frac{\partial}{\partial r} \left( r^{d-1} u^m \frac{\partial p}{\partial r} \right) = 0, \quad p = \frac{1}{r^{d-1}} \frac{\partial u}{\partial r} \quad (29)$$

which is invariant under the scalings

$$t \rightarrow \lambda t, \quad x \rightarrow \lambda^\beta r, \quad u \rightarrow \lambda^{(4\beta-1)/m} u. \quad (30)$$

Again, if the boundary conditions are such that the total mass (17) is conserved, it then follows that  $\beta = 1/(4 + md)$ .

From (30) it follows that for  $m = 1$  there exists a source-type self-similar solution in the closed form

$$u_{SS} = \begin{cases} \left( \frac{t_0}{t} \right)^{d/(4+d)} \left( 1 - \left( \frac{r^2}{K t^{2/(4+d)}} \right) \right)^2 & r^2 \leq K t^{2/(4+d)} \\ 0 & r^2 > K t^{2/(4+d)} \end{cases} \quad (31)$$

where  $t_0, K$  are constants, which may be used to test numerical results. More details of this similarity solution may be found in, for example, [8].

## 5 Numerical Results

### 5.1 The Second Order Problem

Two sets of results are presented to illustrate the accuracy with which the method approximates the second order problem presented in Section 4.1. The first set is in one dimension ( $d = 1$ ). Figure 2 shows that the rate at which the  $L^1$  error in the approximation decreases is roughly proportional to  $(\Delta x)^2$  in the  $m = 1$  case, where  $\Delta x$  is taken to be the length of a cell in the initial (uniform) mesh. The order of accuracy is noticeably lower (approximately 1 for the boundary position and 1.4 for the solution values) when  $m = 2$  because the exact slope of the solution at the boundary is now infinite.

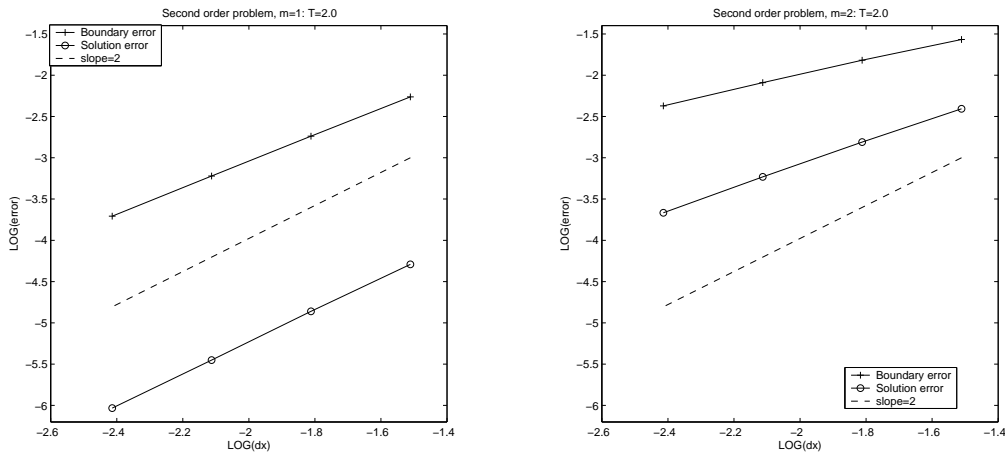


Figure 2: Orders of accuracy in the  $L^1$  norm of approximations to one-dimensional self-similar solutions of the PME for  $m = 1$ ,  $T = t - \frac{1}{8}$  (left) and  $m = 2$ ,  $T = t - \frac{1}{6}$  (right). The dashed line indicates a slope of 2.

Figure 3 shows snapshots of the evolution of a solution given by equation (21) in the cases  $m = 1$  and  $m = 2$  in two dimensions ( $d = 2$ ). It is approximated on a genuinely unstructured, but still uniform, 2349 node, 4539 cell, mesh. Further results, presented in [1, 2], give more details and show that the order of accuracy obtained in two space dimensions is the same as in one.

It should be noted that mass is conserved to machine accuracy in all of these calculations, and indeed also for those presented in the next subsection

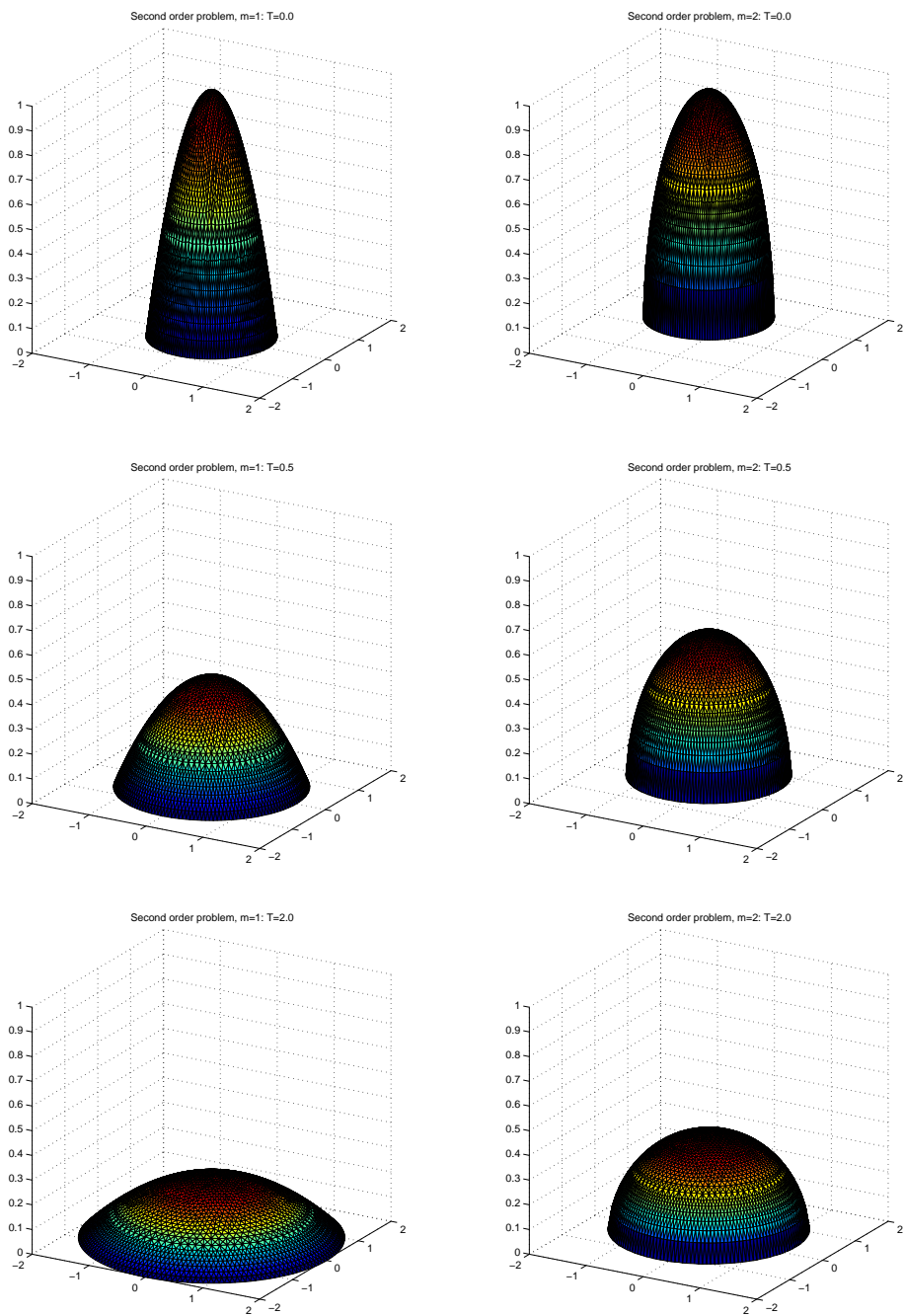


Figure 3: Snapshots of approximations to radially symmetric self-similar solutions (with corresponding triangular meshes) of the PME at three different times for  $m = 1$ ,  $T = t - \frac{1}{8}$  (left) and  $m = 2$ ,  $T = t - \frac{1}{6}$  (right).

for fourth order problems. Furthermore, although the curl of the mesh velocity field is assumed here to be zero, it is possible to successfully impose a background velocity field  $\mathbf{q}$  on the mesh movement equations via the extra vorticity term in (12) [1].

### 5.1.1 Comparison results

The new scheme is not restricted to modelling self-similar solutions.

We have investigated a comparison property of the approximate solutions which reflects the same property of the exact solution of the PME (see (22,23)). This property holds for the approximate solution derived here, as can be seen in Figures 4 and 5 which show two experiments in which the initial conditions are perturbed. In Figure 4 a random perturbation is applied to the initial solution and its evolution compared with two radially symmetric solutions scaled according to the minimum and maximum perturbations. In Figure 5 a sinusoidal perturbation is applied to the initial position of the boundary and its evolution is found to be sandwiched in a similar manner. In both cases the initial random perturbations are smoothed out very rapidly.

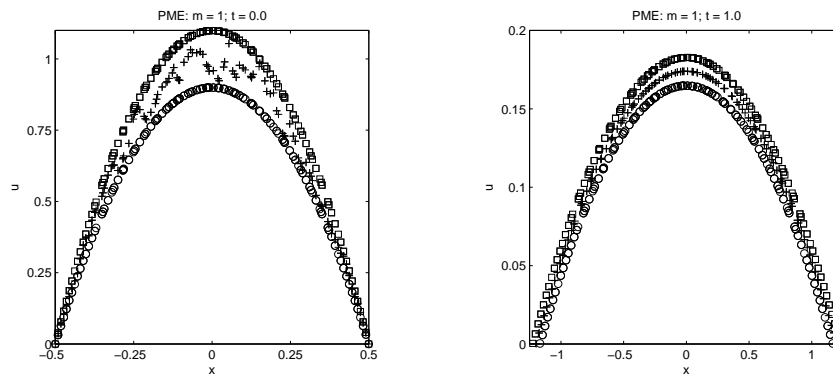


Figure 4: Slices of the initial conditions (left) and approximate solutions at  $t = 1$  (right) taken through the origin, illustrating the ‘sandwiching’ of a randomly perturbed solution to the PME with  $m = 1$ .

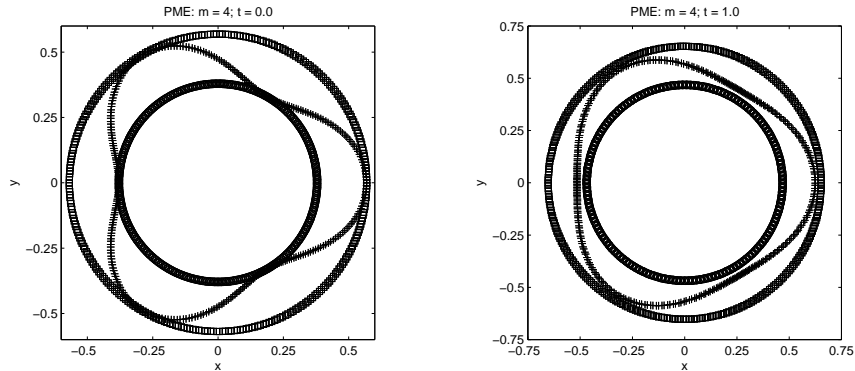


Figure 5: Slices of the initial conditions (left) and approximate solutions at  $t = 1$  (right) taken through the origin, illustrating the ‘sandwiching’ of a sinusoidally perturbed mesh for the PME with  $m = 4$ .

## 5.2 The Fourth Order Problem

Similar sets of results are presented for the fourth order problem described in Section 4.2. Figure 6 shows that the  $L^1$  error now decreases in proportion to  $(\Delta x)^4$  when  $m = 1$  and  $d = 1$  (for which the exact self-similar solution (31) exists).  $\Delta x$  is defined as before.

Figure 7 shows snapshots of the evolution of a solution given by equation (31) in two dimensions ( $d = 2$ ), approximated on a uniform unstructured 545 node, 1024 cell mesh. The accuracy of this approximation is comparable to that obtained in one dimension. Note, though, that explicit time-stepping is being used and finer meshes are very expensive to use because the stability of the method appears to require  $\Delta t$  to reduce in proportion to  $(\Delta x)^4$ .

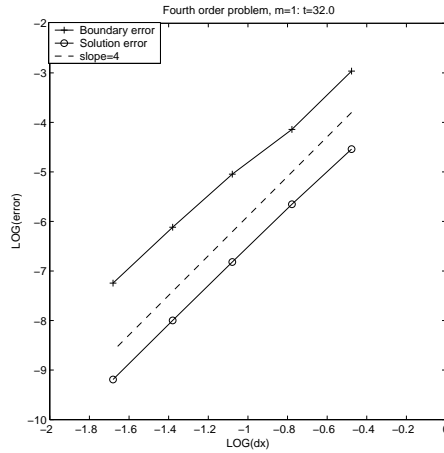


Figure 6: Order of accuracy in the  $L^1$  norm of an approximation to a one-dimensional ( $d = 1$ ) self-similar solution of the fourth order equation with  $m = 1$ .

## 6 Conclusions

A Lagrangian moving finite element method has been described which is based on local mass conservation, in line with the scale invariance of problems that exhibit global mass conservation. The method is illustrated on second order and fourth order nonlinear diffusion problems with moving boundaries for which mass is conserved and analytic self-similar solutions exist. Results show that the method is accurate and exhibits approximate scale invariance.

Self-similar solutions have been considered here for the purpose of illustrating the accuracy of the method, but the method can be applied far more generally. For example, we have shown that in the case of the second order problem the comparison principle (22,23), as well as a similar principle for the boundary, is sustained on a numerical level.

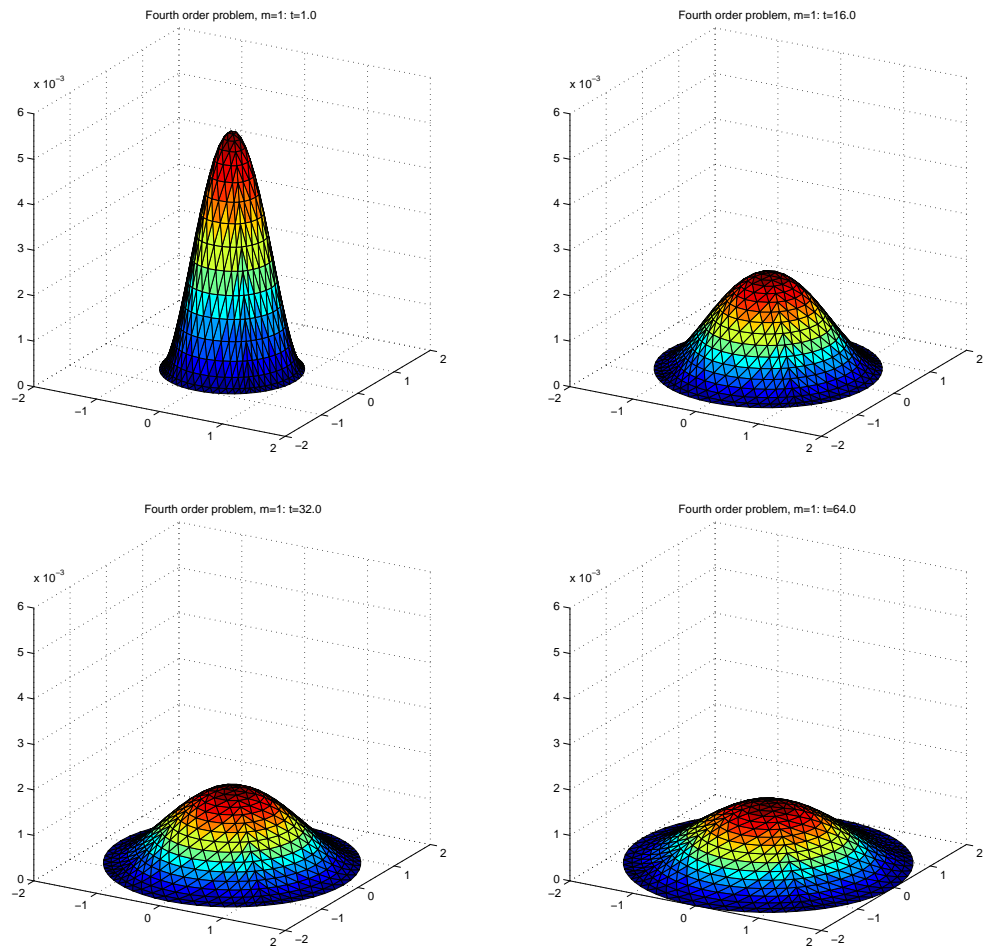


Figure 7: Snapshots of an approximation to a radially symmetric self-similar solution to the fourth order equation at four different times.

## References

- [1] Baines, M.J., Hubbard, M.E. and Jimack, P.K., A Moving Finite Element Method using Monitor Functions, Report 2003.04, School of Computing, University of Leeds, UK (2003).
- [2] Baines, M.J., Hubbard, M.E. and Jimack, P.K., A Lagrangian Moving Finite Element Method using Monitor Functions, in Proceedings of the Third Workshop on Computational Methods in Partial Differential Equations, Hong Kong, (2003).
- [3] Blake, K.W., Moving Mesh Methods for Nonlinear Partial Differential Equations, PhD thesis, Dept of Mathematics, University of Reading, UK (2001).
- [4] Blake, K.W. and Baines, M.J., Moving Mesh Methods for Nonlinear Partial Differential Equations, Numerical Analysis Report 7/01, Dept of Mathematics, University of Reading, UK (2001).
- [5] Budd, C.J, Huang, W. and Russell, R.D., Moving Mesh Methods for Problems with Blow-up, *SIAM J. Sci. Comput.*, 17:305-327 (1996).
- [6] Budd, C.J. and Piggott, M., Geometric Integration and its Applications, *J. Comp. Appl. Math.*, 128:399-422 (2001).
- [7] Cao, W., Huang, W. and Russell, R.D., A Moving Mesh Method based on the Geometric Conservation Law, *SIAM J. Sci. Comput.*, 24:118-142 (2002).
- [8] Diez, J.A., Kondic, L. and Bertozzi, A., Global Models for Moving Contact Lines, *Physical Review E*, 63:011028 (2000).
- [9] Huang, W, Ren, Y. and Russell, R.D , Moving Mesh Partial Differential Equations (MMPDEs) based on the Equidistribution Principle, *SIAM J. Num. Anal.*, 31:709-730 (1994).
- [10] Oleinik, O. et al., *Istv. Akad. Nauk. SSR Ser. Mat.*, 22:667-704 (1958).
- [11] Thomas, P.D. and Lombard, C.K., The Geometric Conservation Law and its Application to Flow Computations on Moving Grids, *AIAA J.*, 17:1030-1037 (1979).